UCSMP Textbook Translations

Japanese Grade 8 Mathematics

Kunihiko Kodaira
Editor

Translator
Hiromi Nagata

Translation Editor
George Fowler

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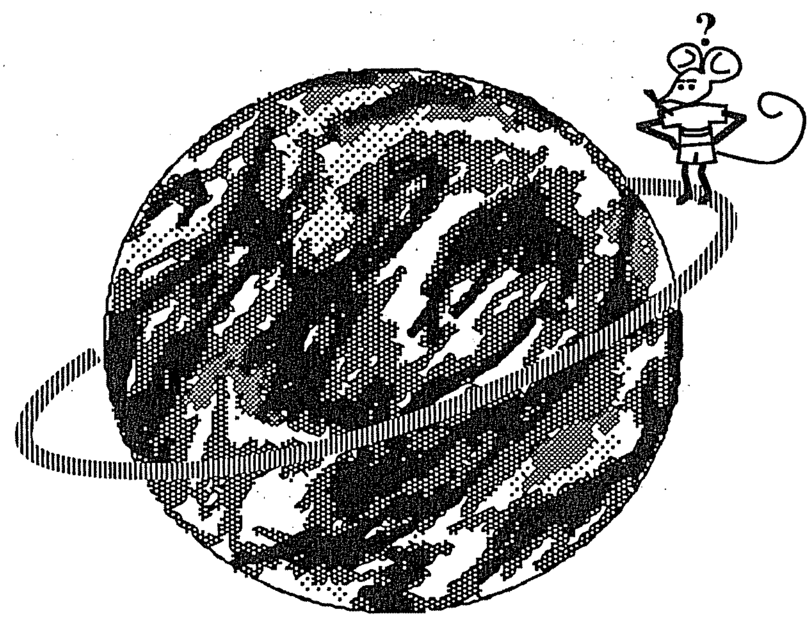
CHAPTER 1

CALCULATING EXPRESSIONS

Imagine the Earth as a very large ball. If we were to circle the equator with a string 10 meters longer than the circumference of the Earth at the equator, how much space would there be between the Earth and the string? Could a mouse pass through this space? The radius of the Earth is 6,378,137 m, but it is much easier to solve this problem by using a letter to represent the radius of the Earth.

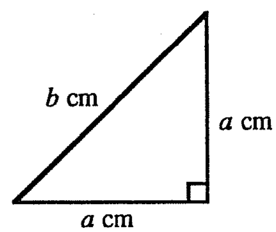
In this chapter we will learn to calculate expressions involving letters. Let's see if we can solve the problem above in the course of the chapter.



1 MONOMIALS AND POLYNOMIALS

1 Monomials and Polynomials

Problem 1 In the diagram on the right, the two sides that form the right angle of a triangle are each a cm long, and the other side is b cm. Write expressions to represent the area and the perimeter of this triangle.



One of these expressions, $\frac{1}{2}a^2$, involves only multiplication, and the other, $2a + b$, involves addition and multiplication.

An expression which involves only multiplication of numbers and letters, such as $2a$ or $\frac{1}{2}a$, is called a **monomial.** A single letter or number, such as b or -5, can also be a monomial.

An expression which involves the sum of monomials, such as $2a + b$ or $3a^2 + 1$, is called a **polynomial.** Each monomial within a polynomial is called a **term.**

Example 1 (1) $3x + 5y$ is a polynomial, and $3x$ and $5y$ are its terms.

(2) $3x^2 - 2x - 5$ can be written as $3x^2 + (-2x) + (-5)$. Therefore, it is a polynomial, and its terms are $3x^2$, $-2x$, and -5.

Problem 2 Which of the following expressions are monomials and which are polynomials?

(1) $4a - b$ (2) $3 - p + q$ (3) $5a^2$

(4) $\frac{1}{4}xy^2$ (5) $5a + b^2 - \frac{1}{3}$

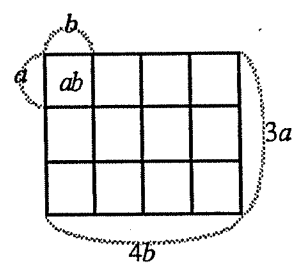
Workbook Exercise 1 →

Multiplication of Monomials

Problem 3 What expression can you use to represent the area of a rectangle which is $3a$ long and $4b$ wide?

A product of monomials, such as $3a \times 4b$, can be calculated using the commutative law and the associative law, as we shall see below.

$$3a \times 4b = 3 \times a \times 4 \times b$$
$$= 3 \times 4 \times a \times b$$
$$= 12ab$$



Problem 4 Check the answer to Problem 3 by looking at the diagram on the right.

Example 2 $8x \times (-4y) = 8 \times x \times (-4) \times y$

$$= 8 \times (-4) \times x \times y$$
$$= -32xy$$

Problem 5 Perform the following calculations:

(1) $x \times 7y$ (2) $6a \times 2b$

(3) $(-3x) \times 5y$ (4) $(-3m) \times (-2n)$

(5) $(-2ab) \times 5c$ (6) $\frac{1}{3}x \times (-6y)$

Example 3 (1) $a \times a^2 = a \times a \times a$
$= a^3$

(2) $(-4m)^2 = (-4m) \times (-4m)$
$= (-4) \times m \times (-4) \times m$
$= (-4) \times (-4) \times m \times m$
$= (-4)^2 m^2$
$= 16m^2$

Problem 6 Perform the following calculations:

(1) $x^2 \times x$ (2) $(-a)^3$

(3) $(-3x)^2$. (4) $5a \times a^2$

Example 4 (1) $-3x^2$ x —(squared)→ x^2 —(times -3)→ $-3x^2$

(2) $(-3x)^2$ x —(times -3)→ $-3x$ —(squared)→ $(-3x)^2 = 9x^2$

Problem 7 Compare the following expressions by writing them in the simplest possible form.

(1) $-5a^2$, $-(5a)^2$, $(-5a)^2$

(2) $-2a^3$, $-(2a)^3$, $(-2a)^3$

Problem 8 Perform the following calculations:

(1) $(-3m)^2 \times 6n$ (2) $(-2a)^2 \times (-3b)$

Division of Monomials

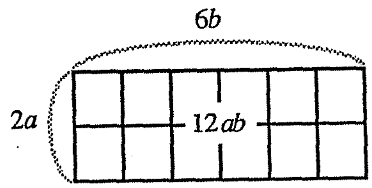
Problem 9 Write an equation to express the length of a rectangle whose width is $6b$ and whose area is $12ab$.

Division problems involving monomials like $12ab \div 6b$ can be calculated in the following way:

$$12ab \div 6b = \frac{12ab}{6b}$$

$$= \frac{\overset{2}{\cancel{12}} \times a \times \overset{1}{\cancel{b}}}{\underset{1}{\cancel{6}} \times \underset{1}{\cancel{b}}}$$

$$= 2a$$



Problem 10 Check the answer above by examining the diagram.

We can simplify fractions that include letters just as we simplify numerical fractions.

Example 5 (1) $8xy \div (-2x)$

$$= -\frac{8xy}{2x}$$

$$= -\frac{\overset{4}{\cancel{8}} \times \overset{1}{\cancel{x}} \times y}{\underset{1}{\cancel{2}} \times \underset{1}{\cancel{x}}}$$

$$= -4y$$

(2) $\frac{1}{2}ab \div \frac{2}{3}a$

$$= \frac{ab}{2} \div \frac{2a}{3}$$

$$= \frac{ab}{2} \times \frac{3}{2a}$$

$$= \frac{\overset{1}{\cancel{a}} \times b \times 3}{2 \times 2 \times \underset{1}{\cancel{a}}}$$

$$= \frac{3}{4}b$$

Problem 11 Perform the following calculations:

(1) $4x \div 2$ (2) $14x \div 7x$

(3) $(-6a) \div 3a$ (4) $2b \div (-6b)$

(5) $2ab \div (-\frac{1}{3}b)$ (6) $4abc \div (-8bc)$

Example 6

(1) $a^2 \div a^3 = \frac{a^2}{a^3} = \frac{\overset{1}{\cancel{a}} \times \overset{1}{\cancel{a}}}{\underset{1}{\cancel{a}} \times \underset{1}{\cancel{a}} \times a} = \frac{1}{a}$

(2) $(-14x^3) \div 2x = -\frac{14x^3}{2x} = \frac{\overset{7}{\cancel{14}} \times \overset{1}{\cancel{x}} \times x \times x}{\underset{1}{\cancel{2}} \times \underset{1}{\cancel{x}}} = -7x^2$

(3) $a^2b \div (-ab^2) = -\frac{a^2b}{ab^2} = \frac{\overset{1}{\cancel{a}} \times a \times \overset{1}{\cancel{b}}}{\underset{1}{\cancel{a}} \times \underset{1}{\cancel{b}} \times b} = -\frac{a}{b}$

Problem 12 Perform the following calculations:

(1) $(-a^3) \div a$ (2) $2x \div x^2$

(3) $12a^2 \div 3a$ (4) $30x^2 \div 5x^2$

(5) $6a^2b \div (-3ab)$ (6) $\frac{2}{3}b^2c \div \frac{5}{6}bc^2$

Calculations Involving Both Multiplication and Division

Calculations involving both multiplication and division are performed as illustrated in the following example.

Example 7

(1) $x^3 \div x^2 \times x$

$$= \frac{x^3 \times x}{x^2}$$

$$= \frac{\cancel{x} \times \cancel{x} \times x \times x}{\cancel{x} \times \cancel{x}}$$

$$= x^2$$

(2) $ab \times b \div a^2 b$

$$= \frac{ab \times b}{a^2 b}$$

$$= \frac{\cancel{a} \times \cancel{b} \times b}{\cancel{a} \times a \times \cancel{b}}$$

$$= \frac{b}{a}$$

Problem 13 Perform the following calculations:

(1) $a \times a^2 \div a$

(2) $a^2 b \div ab^2 \times 3b$

(3) $4x^3 \div (-2x) \div x$

(4) $(-2b) \times a \div b^2$

Exercises

1. Perform the following calculations:

(1) $2x \times (-4x)$

(2) $9a^2 \times (-7a)$

(3) $(-5x)^2 \times x$

(4) $3a \div a$

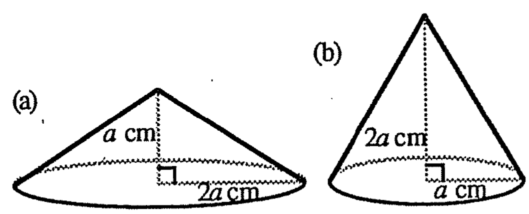
(5) $(-xy) \div xy$

(6) $4m^2 \div m^3$

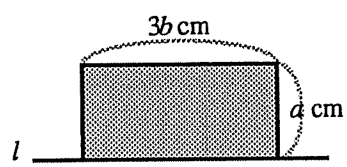
(7) $\frac{1}{12}x^2y \div (-\frac{1}{3}xy^2)$

(8) $xy^2 \div x^2y \times 2x$

2. In the diagram of two cones at the right, what is the relation between the volume of cone (a) and the volume of cone (b)?



3. Find the volume of the solid which is created by revolving the shaded rectangle in the diagram to the right about the axis formed by line l.



2 CALCULATING POLYNOMIALS

1 Polynomials

Degrees of Expressions

Let's consider the monomial $4x^2$, in which x is multiplied by itself: x occurs twice as a factor. The number of times a letter occurs as a factor in a monomial is called the **degree** of the monomial.

The polynomial $x^3 + 4x^2 + 5x$ contains one third degree term, x^3, one second degree term, $4x^2$, and one first degree term, $5x$.

$x^3 = x \times x \times x$ .... third degree
$4x^2 = 4 \times x \times x$ .... second degree
$5x = 5 \times x$ .... first degree

In general, the **degree** of a polynomial is the same as the highest degree of any of its terms. A first degree expression is commonly called a **linear** expression, and a second degree expression is called a **quadratic** expression.

Example 1 $2x$, $-3x + 1$, and $x + y$ are linear expressions.
x^2, $4x^2 - x + 3$, and $1 + y + \frac{3}{5}y^2$ are quadratic expressions.

A monomial can be regarded as a special polynomial containing only one term.

Problem 1 What degree are the following polynomials?

(1) $-4x + 3y$ (2) $2x^2 - 3x + 5$

(3) $3 + x + 5x^2$ (4) $\frac{1}{3}y^2$

Note: The expression a^2b is equivalent to $a \times a \times b$, and these three factors mean that it is a third degree expression. We can also consider the part involving a to be quadratic and the part involving b to be linear.

Combining Like Terms

We learned in grade 7 that terms which have exactly the same letter component are called like terms, and that we can combine like terms by virtue of the distributive law.

$$ac + bc = (a + b)c$$

$$ac - bc = (a - b)c$$

like terms

$5x + 3y + 2x - 7y$

like terms

For example,

$$5x + 3y + 2x - 7y = \underline{5x + 2x} + \underline{3y - 7y}$$

$$= \underline{(5 + 2)x} + \underline{(3 - 7)y}$$

$$= 7x - 4y$$

Example 2 (1) $\frac{3}{2}ab - ab = (\frac{3}{2} - 1)ab$

$$= \frac{1}{2}ab$$

(2) $-5a^2 - 4a + 3a + 7a^2 = -5a^2 + 7a^2 - 4a + 3a$

$$= 2a^2 - a$$

Note: $2a^2$ and $-a$ are not like terms.

Problem 2 Simplify the following expressions by combining the like terms.

(1) $3y - 5y$ (2) $-5b + b$

(3) $\frac{a}{4} - \frac{a}{2}$ (4) $9a - 8 - 6a + 2$

(5) $8a - 7b - a + b$ (6) $xy + \frac{1}{2}xy - 3xy$

(7) $x^2 - xy - x^2 - 2xy$ (8) $a^2 - 2a + 9 + 6a^2 - 4a$

2 Addition and Subtraction of Polynomials

Addition

In grade 7 we learned about addition and subtraction of linear expressions with only one letter.

Problem 1 Calculate $(3x + 2) + (4x - 5)$.

When adding polynomials, we can add together all the terms they contain. First we should combine the like terms.

Example 1 $(3a + 5b - 4c) + (2a - 3b - c)$

$= 3a + 5b - 4c + 2a - 3b - c$

$= (3 + 2)a + (5 - 3)b + (-4 - 1)c$

$= 5a + 2b - 5c$

Adding polynomials vertically, as illustrated at the right for Example 1, allows us to line up the like terms and combine them immediately.

(2)
$$\begin{array}{r} 3a + 5b - 4c \\ +)\ 2a - 3b - c \\ \hline 5a + 2b - 5c \end{array}$$

Problem 2 Perform the following calculations:

(1) $x + (2x - 3y)$ (2) $a - b + (2a - 3b)$

(3) $m - 2n + (3m + 4n)$ (4) $(5x + 6y) + (x - 3y)$

(5) $(4x + 7y + 3) + (-2x + y - 4)$

(6) $(-5m - 9 - 3n) + (6 + 5m - 8n)$

(7) $(3a^2 - 2ab - 4b^2) + (2a^2 + 2ab - 5b^2)$

Problem 3 Add the following pairs of expressions:

(1) $2x - 5y,\ 4x$ (2) $-3x + 7y,\ -8y$

(3) $6a - 4b,\ 2a + 5b$ (4) $a - 3b,\ -2a + 4b$

(5) $3x - 5y - 2,\ -4x + 3y - 5$

(6) $5x^2 + 7x - 6,\ -3x^2 + 2x + 9$

Problem 4 Perform the following calculations:

(1)
$$\begin{array}{rrr} & a & + \ 2b \\ +) & a & - \ b \\ \hline \end{array}$$

(2)
$$\begin{array}{rrrr} & 2x^2 & + \ 3x & \\ +) & -x^2 & + \ 2x & - \ 9 \\ \hline \end{array}$$

Subtraction

Problem 5 Calculate $(4x - 3) - (5x - 7)$.

If we need to subtract one polynomial from another, we can instead add them by reversing the sign of each term in the subtrahend.

Example 2
$$\begin{aligned}(7a - 2c) - (5a - 6b) &= 7a - 2c + (-5a + 6b) \\ &= 7a - 2c - 5a + 6b \\ &= 2a + 6b - 2c\end{aligned}$$

Problem 6 Check the answer to Example 2 by adding $5a - 6b$ to it.

We can also calculate Example 2 by the vertical method, as illustrated below.

$$\begin{array}{rrrr} & 7a & & -\ 2c \\ -) & 5a & -\ 6b & \\ \hline & 2a & +\ 6b & -\ 2c \end{array} \qquad \begin{array}{rrrr} & 7a & & -\ 2c \\ +) & -5a & +\ 6b & \\ \hline & 2a & +\ 6b & -\ 2c \end{array}$$

Problem 7 Perform the following calculations:

(1) $x - (2x - 3y)$ (2) $a - 3b - (2a - 3b)$

(3) $m - 2n - (3m + 5n)$ (4) $(5x + 6y) - (x - 3y)$

(5) $(-2x + 8y) - (x - 2y + 3)$ (6) $(x^2 - 3x + 4) - (x^2 - x + 5)$

(7) $(2a - 3b + 5c) - (-3a - 7c)$

Problem 8 Subtract the second expression from the first in each of the following pairs:

(1) $2x - 5y$, $3x$

(2) $-3x - 7y$, $-8y$

(3) $6a - 4b$, $2a + 5b$

(4) $a - 3b$, $-2a + 4c$

(5) $3x - 5y - 2$, $-4x + 3y - 5$

(6) $5x^2 + 7x - 6$, $-3x^2 + 2x +$
9

Problem 9 Perform the following calculations:

(1)
$$\begin{array}{rrr} & 4x & - \ 5y \\ -) & -x & + \ 3y \\ \hline \end{array}$$

(2)
$$\begin{array}{rrrr} & -5x^2 & & - \ 7 \\ -) & 3x^2 & - \ 2x & + \ 5 \\ \hline \end{array}$$

Drills

1. Perform the following calculations:

(1) $4x + 3x$ (2) $5a^2 - 3a^2$

(3) $-3m + 8m - 7m$ (4) $-6y^2 + y + 5y^2$

(5) $2a + (3a - 5b)$ (6) $4x - (3y - 2x)$

2. Perform the following calculations:

(1) $(4x - 5y) + (x + 4y)$

(2) $(3a - b) - (-2a + 4b)$

(3) $(-5x + 9y) + (x - 3y - 5)$

(4) $(7x - 5y) - (-3x + 2y - 6)$

(5) $(2x^2 - 9x + 5) + (7x^2 - 6x - 4)$

(6) $(-3x^2 + 7x - 8) - (5x^2 - 4x + 3)$

(7) $(a^2 - 3ab - 6b^2) + (-4a^2 + 2ab + 7b^2)$

(8) $(6a^2 + 2ab - b^2) - (-2a^2 + 3ab + b^2)$

3 Multiplication and Division of Polynomials and Monomials

Multiplication

Problem 1 Remove the parentheses in the following expressions:

(1) $4(x + 2)$ (2) $-2(3x - 5)$

The distributive law can be used to multiply monomials and polynomials, as illustrated below.

$$a(b + c - d) = ab + ac - ad$$

Example 1 $-5(3x - y + 2) = (-5) \times 3x - (-5) \times y + (-5) \times 2$
$= -15x + 5y - 10$

Problem 2 Perform the following calculations:

(1) $3(a + 4b)$ (2) $-2(x - 7y)$

(3) $-(2x - 10y)$ (4) $2(3a - 5b + 1)$

Example 2 (1) $2a(3a - 5b) = 2a \times 3a - 2a \times 5b$

$$= 6a^2 - 10ab$$

(2) $-3x(x - 2y + 5)$

$$= (-3x) \times x - (-3x) \times 2y + (-3x) \times 5$$

$$= -3x^2 + 6xy - 15x$$

Problem 3 Perform the following calculations:

(1) $4a(a + 3b)$ (2) $b(5a - b)$

(3) $-5l(2l - 7m)$ (4) $-6p(-p + 2q - 5)$

Division

We can also use the distributive law to divide polynomials by monomials, as illustrated below.

$$(a + b) \div c = (a + b) \times \frac{1}{c}$$

$$= \frac{a}{c} + \frac{b}{c}$$

Thus,

$$\frac{a + b}{c} = \frac{a}{c} + \frac{b}{c}$$

Example 3 (1) $(6a - 9b) \div 3$

$= \dfrac{6a - 9b}{3}$

$= \dfrac{6a}{3} - \dfrac{9b}{3}$

$= 2a - 3b$

(2) $(x^2y + xy^2) \div xy$

$= \dfrac{x^2y + xy^2}{xy}$

$= \dfrac{x^2y}{xy} + \dfrac{xy^2}{xy}$

$= x + y$

Problem 4 Perform the following calculations:

(1) $(12x - 20y) \div 4$ (2) $(-9a + 18b) \div (-3)$

(3) $(a^3 - 5a) \div a$ (4) $(6l^2m - 9lm^2) \div (-3lm)$

Calculations Involving the Four Arithmetic Operations

Problem 5 Calculate $5(x - 2) + (2x + 3)$.

Example 4 (1) $4\,(2x + y) - 3\,(x - 5y)$

$= 8x + 4y - 3x + 15y$

$= 5x + 19y$

(2) $2x\,(x + 3) - x\,(x - 2)$

$= 2x^2 + 6x - x^2 + 2x$

$= x^2 + 8x$

Problem 6 Perform the following calculations:

(1) $3\,(x - 2y) - 5\,(4x + y)$

(2) $5\,(a - b) - 2\,(2a - 3b)$

(3) $3x\,(x - 4) + 2x\,(x + 5)$

(4) $2a\,(a - 3) - 3a\,(a - 4)$

Example 5 Let's calculate $\frac{3x-y}{2} - \frac{x-4y}{4}$.

(1) $\frac{3x-y}{2} - \frac{x-4y}{4}$

$= \frac{2(3x-y) - (x-4y)}{4}$

$= \frac{6x-2y-x+4y}{4}$

$= \frac{5x+2y}{4}$

(2) $\frac{3x-y}{2} - \frac{x-4y}{4}$

$= \frac{1}{2}(3x-y) - \frac{1}{4}(x-4y)$

$= \frac{3}{2}x - \frac{1}{2}y - \frac{1}{4}x + y$

$= \frac{5}{4}x + \frac{1}{2}y$

Note: In Example 5, you may calculate by either method [1] or method [2].

Problem 7 Perform the following calculations:

(1) $\frac{5x-y}{3} - \frac{3x+y}{2}$ (2) $\frac{2x+y}{3} - \frac{x-2y}{6}$

(3) $\frac{2}{3}a + \frac{1}{3}(a-b)$ (4) $m - n - \frac{m-2n}{3}$

Drills 1. Perform the following calculations:

(1) $6(5a+7b)$ (2) $-2a(x-8y)$

(3) $-(x-y+z)$ (4) $-\frac{a}{3}(2p-3q-6)$

(5) $(4x-6y) \div 2$ (6) $(6a^2-2a) \div 2a$

(7) $(9a^2b-15ab) \div 3ab$ (8) $(16pq-8p^2) \div (-4p)$

2. Perform the following calculations:

(1) $2x-3y-2(3x-2y)$ (2) $3a(a-2b)-2(ab-3)$

(3) $\frac{y}{2} + \frac{x-y}{3}$ (4) $2x-3y-\frac{3x-5y}{2}$

4 Practical Applications of Calculating Expressions

Calculating the Value of Expressions

Problem 1 Find the value of the following expression for $x = 4, y = -\frac{1}{5}$:

$$2xy - 7xy$$

When finding the value of an expression by substituting numbers for the letters, it is sometimes easier to calculate it if we first simplify the expression and then substitute.

Example 1 Find the value of the following expression for $x = 2, y = -\frac{1}{3}$:

$$(12x^2y - 9xy^2) \div 3xy$$

[Solution] $(12x^2y - 9xy^2) \div 3xy = 4x - 3y$

Here, if we substitute in $x = 2, y = -\frac{1}{3}$, we get

$$4x - 3y = 4 \times 2 - 3 \times \left(-\frac{1}{3}\right)$$
$$= 8 + 1$$
$$= 9$$

Answer: 9

Problem 2 Find the value of the following expressions for $a = \frac{1}{2}$, $b = 3$:

(1) $5a^2b \div ab \times 2b$ (2) $(7a - 8b) - (11a - 5b)$

(3) $(2a^2 - 3ab) \div (-a)$ (4) $3a(a - 2b) + a(a + 4b)$

Problem 3 A rectangle is twice as long as it is wide. Consider a rectangle 1 cm wider than the original rectangle and another rectangle 1 cm longer than the original rectangle. Which rectangle has a greater area? How much greater? Take the width of the original rectangle to be first 57 cm, and then 125 cm.

Demonstrating the Properties of Numbers by Expressions

We can explain why certain properties of numbers hold true by calculating expressions.

Example 2 The sum of the different multiples of 4 is also a multiple of 4. Let's explain why this is so.

We can represent the two multiples of 4 by the following expressions, assuming that m and n are integers.

$$4m, \qquad 4n$$

Using the distributive law, we can rewrite their sum:

$$4m + 4n \quad = \quad 4(m + n)$$

Since m and n are integers, their sum

$$m \quad + \quad n$$

is also an integer. Therefore,

$$4(m \quad + \quad n)$$

is a multiple of 4.

Hence, the sum of the two multiples of 4 is also a multiple of 4.

Problem 4 The sum of two multiples of a natural number a is itself a multiple of a. Demonstrate that this is true as in Example 2.

Example 3 The sum of the individual digits of 648, $6 + 4 + 8$, is a multiple of 9. In fact, any natural number is a multiple of 9 if the sum of its digits is a multiple of 9. Let's explain why this is so for a three-digit natural number.

Let us take x as the digit in the hundreds place, y as the digit in the tens place, and z as the digit in the ones place. Now we can represent the three-digit natural number as

$$100x + 10y + z$$

We can then transform this expression into a form we can use:

$$\begin{aligned} 100x + 10y + z &= (99 + 1)x + (9 + 1)y + z \\ &= 99x + x + 9y + y + z \\ &= 99x + 9y + x + y + z \\ &= 9\,(11x + y) + (x + y + z) \end{aligned}$$

Since

$$11x + y$$

is an integer,

$$9(11x + y)$$

is a multiple of 9.

The sum of the individual digits

$$x + y + z$$

is also a multiple of 9. Therefore,

$$100x + 10y + z$$

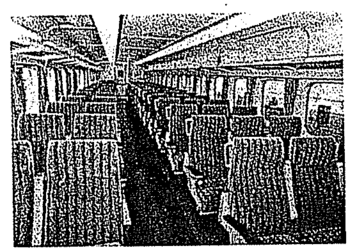
is the sum of two multiples of 9, and so it is also a multiple of 9. From this demonstration, we can see that if the sum of the digits of any natural number is 9, the number itself must also be a multiple of 9.

Problem 5 If the sum of the digits of a natural number is 3, then that number is also a multiple of 3. Demonstrate that this is true for a three-digit number.

Transforming Equations

Calculating expressions can also help us in transforming equations.

Problem 6 The seats in the Japanese bullet train are arranged in rows of 5 seats, two on one side of the aisle and three on the other. We reserved x rows of two seats each and y rows of three seats each for a group of 48 people.



(1) Write an equation to express the relation between x and y.

(2) Use the equation you wrote in (1) to find the value of y if x is 6; if x is 9.

To find the value of y for given values of x in the equation $2x + 3y = 48$, it is convenient first to transform the equation so that y is expressed in terms of x.

Example 4 Let's transform the following equation to express y in terms of x.

$$2x + 3y = 48 \qquad (1)$$

Transposing $2x$,

$$3y = -2x + 48$$

Dividing both sides by 3,

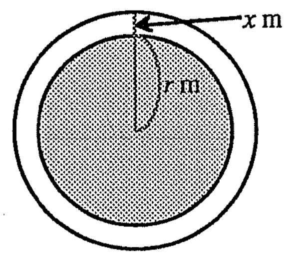
$$y = -\frac{2}{3}x + 16 \qquad (2)$$

Transforming an equation like (1) to express y in terms of x is called solving the equation for y.

Problem 7 Solve the following equations for the letters given in brackets:

(1) $5x + y = 10$ [x] (2) $2x - 3y = 1$ [y]

Example 5 Imagine the Earth as a very large ball with a radius of r m. We circle the Earth with a string a m longer than the circumference of the Earth at the equator. If we now take x m as the distance between the Earth and the string, then



$$2\pi(r+x) = 2\pi r + a$$

Let's transform this equation so that x is expressed in terms of a. Removing the parentheses and rearranging the terms, we arrive at

$$2\pi x = a$$

$$x = \frac{a}{2\pi}$$

Problem 8 If we circle the Earth's equator with a string 10 cm longer than the circumference of the Earth at the equator, what is the distance between the Earth and the string? Take 3.14 as the value of π, and calculate the answer to two decimal places.

Example 6 The formula below enables us to find the area S of a trapezoid with an upper base of a, a lower base of b, and a height of h.

$$S = \frac{1}{2}(a+b)h \qquad (1)$$

Transform the formula so that it gives us the lower base in terms of the upper base, the height, and the area.

[Solution] If we multiply both sides of (1) by 2 and transpose the two sides, we get

$$(a+b)h = 2S$$

Dividing both sides by h,

$$a+b = \frac{2S}{h}$$

Transposing a,

$$b = \frac{2S}{h} - a$$

Answer: $b = \frac{2S}{h} - a$

Problem 9 Solve the following equations for the letters given in brackets:

(1) $S = ab$ [b] (2) $V = \frac{1}{3}Sh$ [S]

(3) $180 = a + b + c$ [a] (4) $l = 2(a + b)$ [a]

Exercises

1. Perform the following calculations:

(1) $-3a - 2a + 4a$ (2) $3x - 2y - (2x - 5y)$

(3) $2(a - b) - 5(2a - b)$ (4) $(2x - 3y)x - 4xy$

(5) $(-8a + 4ab) \div 2a$ (6) $\frac{4x - 3y}{2} - \frac{x + 2y}{3}$

2 Solve the following equations for the letters given in brackets:

(1) $6x - 2y = 5$ [y] (2) $l = 2a + 2\pi r$ [r]

(3) $l = \frac{\pi ra}{180}$ [a] (4) $a = b(1 + c)$ [c]

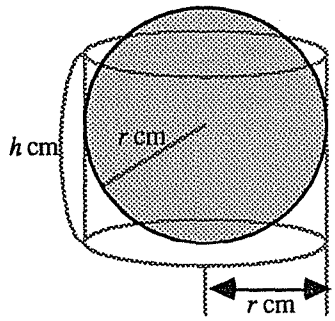
3. If m and n are integers, the expressions $2m + 1$ and $2n + 1$ represent two odd numbers. Use this fact to demonstrate that the following generalization is true:

(odd number) + (odd number) = (even number)

4. A ball of modeling clay with a radius of r cm is reworked into the form of a cylinder with a base radius of r cm and an altitude of h cm.

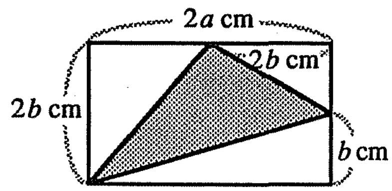
(1) Write an equation to express the fact that the ball and the cylinder have equal volumes.

(2) If the radius of the ball is 15 cm, what is the altitude of the cylinder?



5. Take a two-digit natural number which does not have 0 in the ones place, and make a new number by reversing the digits in the original number. If we add these two numbers together, the sum will be a multiple of 11. Explain why this is so.

6. Find the shaded area in the rectangle to the right.



Chapter Exercises

A

1. Perform the following calculations:

(1) $7a \times b$

(2) $5x \times (-3x)$

(3) $(-a)^2 \times 4a$

(4) $9x^2 \div x$

(5) $8ab \div 4a^2$

(6) $5x^2y \div \frac{x}{3}$

(7) $2ab \times (-6b) \div (-4ab^2)$

(8) $4x^2 \div (-x)^3 \times \frac{1}{2}x$

(9) $3a(2b - 3)$

(10) $(a + 4b) \times (-ab)$

(11) $(2a - 6b) \div 2$

(12) $(9x^2 + 12xy) \div 3x$

2. Perform the following calculations:

(1) $4a + 3a - 5a$

(2) $7x + 2y - 4x - 3y$

(3) $3x^2 - (x + 2 + x^2) + 3$

(4) $(4x - 7y) - (3x - 5y)$

(5) $\frac{2x + y}{3} - \frac{x - y}{2}$

(6) $\frac{3x - 5y}{4} - \frac{x - 2y}{2}$

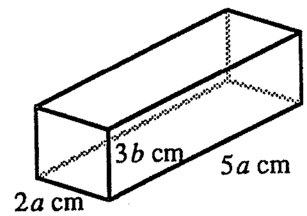
(7) $(3a - 5b + 2c) - (-2a - 3b + 4c)$

3. Find the value of the following expressions for $a = 5$, $b = -3$:

(1) $4a + 5b - 2a + b - 8$

(2) $(6a^2b + 3ab^2) \div 3ab$

4. Find the surface area of a rectangular parallelepiped $5a$ cm long, $2a$ cm wide, and $3b$ cm high.



5. If we double the radius of a circle, its area increases by a factor of 4. Explain why this is so.

6. Solve the following equations for the letters given in brackets:

(1) $V = \pi r^2 h$ [h] (2) $S = \frac{1}{2}lr$ [l]

B

1. Perform the following calculations:

(1) $a^2 + (a^3 + a)$ (2) $(-\frac{1}{2}x)^3 + (-\frac{1}{4}x)^2$

(3) $(4x^2y - 6xy^2) + \frac{1}{2}xy$ (4) $12\left(\frac{5a - 4b}{3} - \frac{a - 2b}{6}\right)$

(5) $6x^3 + (-2x) + (-2x)^2 \times \frac{5}{2}$ (6) $\frac{5x - 4y}{3} - \frac{x - 2y}{6}$

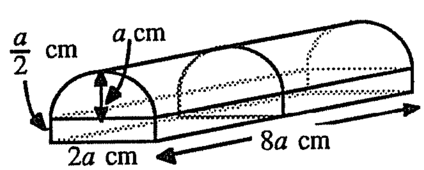
2. Given that $A = x^2 - 3x + 2$, $B = 2x^2 - 1$, and $C = -3x + 2$, perform the following calculations:

(1) $A + B + C$ (2) $A - (B + C)$

3. Let's think of a two-digit number such as 73. If we reverse the digits to make a new natural number, 37, and subtract the smaller number from the larger one, we get 36, which is a multiple of 9. This holds true no matter what the original number is, as long as it does not have a 0 in the ones place. Explain why this is so, taking a as the digit in the tens place and b as the digit in the ones place, and assuming that $a > b$.

$$\begin{array}{r} 73 \\ -)\ 37 \\ \hline 36 \end{array}$$

4. In the figure to the right we have wrapped a string around the middle of a geometric solid formed by placing a half-cylinder on top of a rectangular parallelepiped. Find the following quantities:

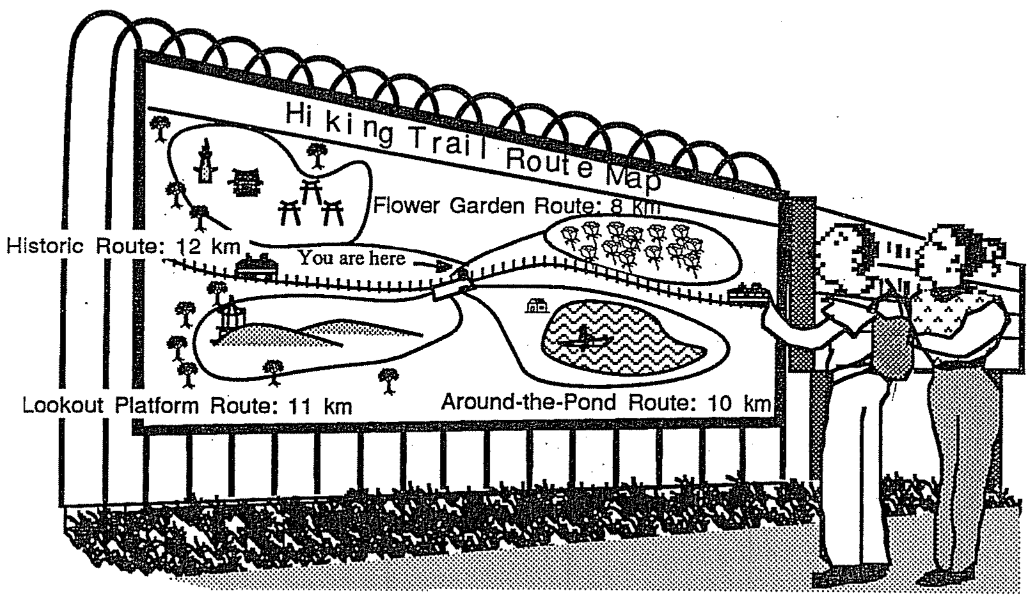


(1) The length of the string. (2) The surface area of the geometric solid.

CHAPTER 2

INEQUALITIES

Four hiking trails start at a train station. A boy and his group want to leave this station at 10 o'clock, walk along one of the routes, and return to the station by 2:30. If they can walk 3 km/hour and will make a one-hour stop for lunch along the way, which route should they take?

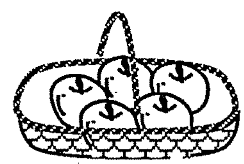


1 INEQUALITIES

1 Inequalities and Their Solutions

Relations among quantities include "greater than" and "less than" as well as "equal to." We express these relations with the inequality signs > and <.

Example 1 If you buy 5 apples at a yen apiece and put them in a basket that costs b yen, and the total expenditure is less than or equal to 1,000 yen, you can write the following relation:



$$5a + b \leq 1{,}000$$

Example 2 The sum of the lengths of two sides of a triangle is greater than the length of the third side. This can be expressed by letting the first two sides equal x cm and y cm, and the third side z cm:

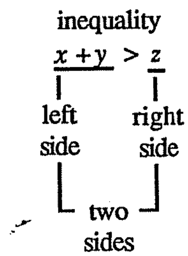
$$x + y > z$$

Problem 1 Express the relations between the following quantities using inequality signs.

(1) If we multiply a number x by 2 and add 5, the result is greater than 10.

(2) When we drive a distance of 50 km at v km/hour, it takes at least t hours.

An expression that involves inequality signs is called an **inequality**. The left part of the inequality is referred to as the left side, the right part as the right side, and the left and the right sides together are referred to as the two sides (or both sides) of the inequality.



Solutions to Inequalities

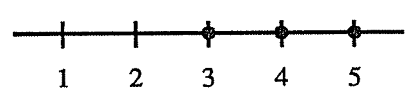
Certain equalities, such as $3x + 4 = 5$, are true only when particular values are substituted for the letters. Is this true of inequalities as well?

Example 3 Let's determine which members of the set $\{1, 2, 3, 4, 5\}$ can be substituted for x while the following inequality remains true:

$$4x + 5 \geq 17$$

value of x	left side	sign	right side	$4x + 5 \geq 17$
1	$4 \times 1 + 5 = 9$	<	17	not true
2	$4 \times 2 + 5 = 13$	<	17	not true
3	$4 \times 3 + 5 = 17$	=	17	true
4	$4 \times 4 + 5 = 21$	>	17	true
5	$4 \times 5 + 5 = 25$	>	17	true

If we substitute the members of the set $\{1, 2, 3, 4, 5\}$ one by one into the inequality, the inequality holds only when the value of x is 3, 4, or 5.



The values of a letter for which an inequality is true are called the **solutions** to the inequality. The solutions to the inequality in Example 3 are 3, 4, and 5, and the set of all the solutions is $\{3, 4, 5\}$. The set of all solutions is called the solution set.

Problem 2 Find the members of the set $\{1, 2, 3\}$ which are solutions to the following inequalities.

(1) $4x + 5 > 12$ (2) $3x - 1 \leq 2$

2 Properties of Inequalities

The properties of equalities enabled us to work with equations, but are there any cases where we cannot find the solution set for an inequality in the same way? Let's first see whether inequalities are governed by properties similar to the properties of equalities.

We can summarize the properties of equalities as follows:

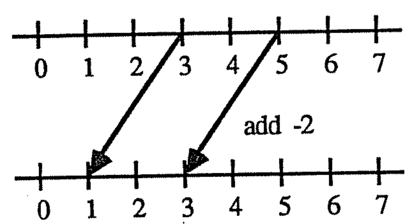
(1) If we add the same number to both sides of an equality, or subtract the same number from both sides, the equality remains valid.

(2) If we multiply both sides of an equality by the same number, or divide both sides by the same number (except 0), the equality remains valid.

Let's see whether or not these properties hold by substituting the word "inequality" for each occurrence of the word "equality."

Example 1 If we add -2 to both sides of
5 > 3,

left side: 5 + (-2) = 3
right side: 3 + (-2) = 1



Because

$$3 > 1,$$

the resulting inequality remains true:

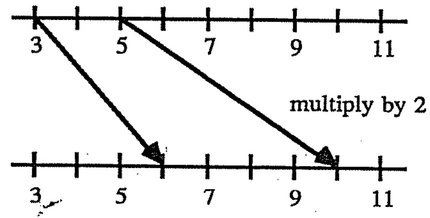
$$5 + (-2) > 3 + (-2)$$

Thus, if we add -2 to both sides of 5 > 3, the direction of the inequality sign does not change.

Problem 1 Subtract -3 from both sides of 5 > 3. Does the direction of the inequality sign change?

Example 2 If we multiply both sides of $5 > 3$ by 2,

left side: $5 \times 2 = 10$
right side: $3 \times 2 = 6$



However, because

$$10 > 6$$

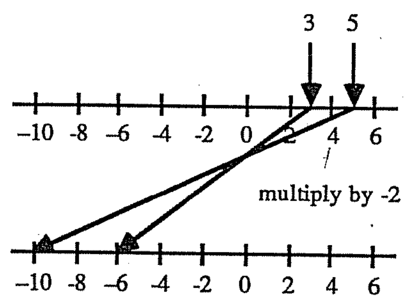
the resulting inequality remains true:

$$5 \times 2 > 3 \times 2$$

Thus, if we multiply both sides of $5 > 3$ by 2, the direction of the inequality sign does not change.

Example 3 If we multiply both sides of $5 > 3$ by -2,

left side: $5 \times (-2) = -10$
right side: $3 \times (-2) = -6$



Because

$$-10 < -6$$

the resulting inequality can only be

$$5 \times (-2) < 3 \times (-2)$$

Thus, when we multiply both sides of $5 > 3$ by -2, the direction of the inequality sign does change.

Problem 2 Multiply both sides of $-2 < -1$ by -5. Does the direction of the inequality sign change?

Problem 3 Divide both sides of $5 > 3$ by 2. Then divide both sides of the original inequality by -2. Does the direction of the inequality sign change?

As we have seen, inequalities have the following properties.

Properties of Inequalities

(1) If we add the same number to both sides of an inequality, or subtract the same number from both sides, the direction of the inequality sign does not change.

If $A > B$, then $A + C > B + C$ and $A - C > B - C$.

(2) If we multiply both sides of an inequality by the same positive number, or divide both sides by the same positive number, the direction of the inequality sign does not change.

If $A > B$ and $C > 0$, then $AC > BC$ and $\frac{A}{C} > \frac{B}{C}$.

(3) If we multiply both sides of an inequality by the same negative number, or divide both sides by the same negative number, the direction of the inequality sign changes.

If $A > B$ and $C < 0$, then $AC < BC$ and $\frac{A}{C} < \frac{B}{C}$.

Problem 4 $A \geq B$ expresses the relation "$A > B$ or $A = B$." Put the appropriate inequality sign in the following [ ].

(1) If $A \geq B$, then $A + C$ [ ] $B + C$, $A - C$ [ ] $B - C$

(2) If $A \geq B$ and $C > 0$, then AC [ ] BC, $\frac{A}{C}$ [ ] $\frac{B}{C}$

(3) If $A \geq B$ and $C < 0$, then AC [ ] BC, $\frac{A}{C}$ [ ] $\frac{B}{C}$

Problem 5 In (1)–(3) below, the properties of inequalities allow you to transform the upper inequality into the lower inequality, and vice versa. Explain how each inequality is transformed.

(1) $\begin{cases} a + 7 < 5 \\ a < 5 - 7 \end{cases}$ (2) $\begin{cases} 2y > 8 \\ y > 4 \end{cases}$

(3) $\begin{cases} -4x \leq 8 \\ x \geq -2 \end{cases}$

3 Solving Linear Inequalities

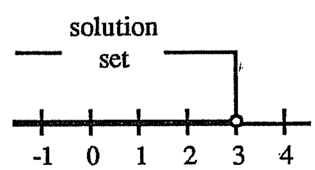
In solving inequalities, if the range of numbers considered changes, the solution set sometimes changes as well.

Problem 1 Find all the values of x from the set of natural numbers 5 or less $\{1, 2, 3, 4, 5\}$ such that the following inequality holds:

$$x - 4 < 2 - x \qquad (1)$$

Problem 2 If we take as the range of values the set of all integers, what is the solution set for inequality (1)?

If we extend the range to all numbers, the solution to (1) becomes all numbers less than 3.



In problems like this one, it is impossible to find the solution by substituting all potential values into the inequality one by one. In the following example we will see how to find a solution set by using the properties of inequalities.

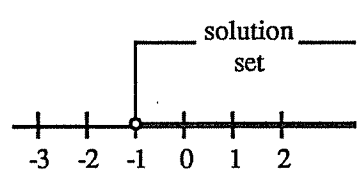
Example 1 Let's find the solution set for the inequality

$$x + 5 > 4 \qquad (1)$$

If we subtract 5 from both sides,

$$x + 5 - 5 > 4 - 5$$

$$x > -1$$



Even if we substitute numbers greater than -1 for x in (1), inequality (1) is still valid. In other words, the solution set for (1) is the set of all numbers greater than -1.

The set of all numbers greater than -1 is expressed as

$$\{x \mid x > -1\}$$

This is the solution set for (1).

Finding the solution set for an inequality is called **solving an inequality.**

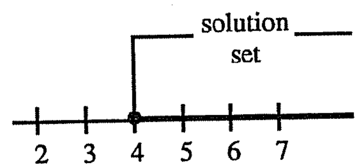
Problem 3 Solve the following inequalities:

(1) $x - 4 > -3$ (2) $5 + x < -7$ (3) $x + 8 \geq 8$

Example 2 Let's solve $3x \geq 12$. If we divide both sides by 3, we get

$$\frac{3x}{3} \geq \frac{12}{3}$$

$$x \geq 4$$



The solution set is

$$\{x \mid x \geq 4\}.$$

Problem 4 Solve the following inequalities:

(1) $4x < 16$ (2) $14 < 7x$

(3) $\frac{3}{4}x \leq 12$

Note: (2) is easier to solve if you first transpose the left and right sides: $7x > 14$.

In Example 2, we expressed the solution set as

$$\{x \mid x \geq 4\}$$

but from now on we will write the conditions which define a set in the form

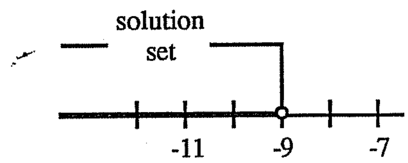
$$x \geq 4.$$

Example 3 Solve $-4x > 36$.

[Solution] If we divide both sides by -4, we get

$$\frac{-4x}{-4} < \frac{36}{-4}$$

$$x < -9$$



Answer: $x < -9$

Problem 5 Solve the following inequalities:

(1) $-5x > 20$ (2) $-2x > 7$

(3) $-3x \leq -21$

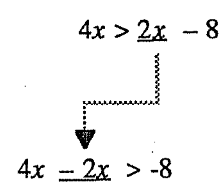
Inequality property (1) allows us to transpose terms in inequalities, just as in equalities.

Example 4 Solve $4x > 2x - 8$.

[Solution] If we transpose $2x$ from the right side to the left side, we get

$$4x - 2x > -8$$

Simplifying the left side,



$$2x > -8$$

If we divide both sides by 2,

$$x > -4$$

Answer: $x > -4$

Problem 6 Solve the following inequalities:

(1) $2x < 3 + x$ (2) $2x - 8 < 14$

(3) $3x \geq 4x + 5$

An inequality that can be transformed into any of the following forms by transposing and simplifying terms is called a **linear inequality**.

(linear expression) > 0, (linear expression) < 0,

(linear expression) ≥ 0, (linear expression) ≤ 0,

For example, because the inequality in Example 4 can be transformed into $2x + 8 > 0$, it is a linear inequality.

Example 5 Solve $4x - 3 > 7x + 9$.

〚 Approach 〛 Collect the terms that include x on the left side and the numerical terms on the right side.

[**Solution**]

If we transpose -3, $4x > 7x + 9 + 3$

Transposing $7x$ and simplifying, $4x - 7x > 9 + 3$

$-3x > 12$

If we divide both sides by -3, then $x < -4$

Answer: $x < -4$

Problem 7 Solve the following inequalities:

(1) $2x + 3 > x - 4$ (2) $4x - 7 \geq 5x - 3$

(3) $7x - 8 > x - 10$ (4) $2x - 6 \leq 6x + 4$

Example 6 Solve $7x - 2(x - 3) < 16$.

〚 Approach 〛 The easiest way to solve this problem is to start by removing the parentheses.

Problem 8 Solve the inequality in Example 6.

Problem 9 Solve the following inequalities:

(1) $2(x-1) < 4$ (2) $2x-(3x-4) \geq 6$

(3) $x-(4x-1) < -5$ (4) $5-2(x+3) > -3$

Example 7 Solve $0.2x + 3.5 < 0.6x - 1.3$.

[Approach] First transform the coefficients of x into integers.

[Solution] If we multiply both sides by 10, then

$$2x + 35 < 6x - 13$$

$$2x - 6x < -13 - 35$$

$$-4x < -48$$

$$x > 12$$

Answer: $x > 12$

Example 8 Solve $\frac{x-1}{3} - \frac{2x+5}{4} > -2$.

[Approach] Transform the inequality so that it contains no fractions.

[Solution] If we multiply both sides by 12, then

$$\frac{x-1}{3} \times 12 - \frac{2x+5}{4} \times 12 > (-2) \times 12$$

$$4(x-1) - 3(2x+5) > -24$$

$$4x - 4 - 6x - 15 > -24$$

$$4x - 6x > -24 + 4 + 15$$

$$-2x > -5$$

$$x < \frac{5}{2}$$

Answer: $x < \frac{5}{2}$

Problem 10 Solve the following inequalities:

(1) $3.1x - 4.2 \geq 1.8x - 1.6$ (2) $4.8 + x < 3.4x$

(3) $\frac{x-2}{3} > \frac{x+1}{4}$ (4) $\frac{2}{3}x + \frac{1}{2} \leq \frac{3}{4}x$

As you have seen, solving linear inequalities is very similar to solving linear equations. One difference is that if we multiply or divide both sides of an inequality by a negative number, the direction of the inequality sign changes.

Drills

1. Solve the following inequalities:

(1) $x + 9 > 4x - 6$ (2) $10 - x \geq -6x$

(3) $2(x - 1) < 7 + 5x$ (4) $3(x - 1) - (x - 5) < x - 3$

(5) $0.8 - 0.2x \leq 0.5x - 0.6$ (6) $\frac{x+2}{6} - \frac{x}{3} \geq x - 3$

4 Applying Linear Inequalities

Example 1 We want to put apples that cost 85 yen apiece in a basket and buy the apples and basket at a total price less than or equal to 1,400 yen. If the basket itself costs 60 yen, how many apples can we buy?

[Approach] Suppose we buy x apples; the total price, including 60 yen for the basket, will be

$$(85x + 60) \text{ yen}$$

[Solution] If we buy x apples, then

$$85x + 60 \leq 1400$$

$$85x \leq 1340$$

$$x \leq 15\frac{13}{17}$$

Because the greatest integer less than or equal to $15\frac{13}{17}$ is 15, we can buy up to 15 apples.

Answer: up to 15 apples

Problem 1 We want to make up a gift that costs less than or equal to 1,000 yen by putting one 50-yen eraser and some 30-yen pencils in a 600-yen pencil case. How many pencils can we put in the pencil case?

Example 2 There are two containers, A and B. A contains 18 liters of water and B contains 3 liters. We pour a certain amount of water from A into B, but A still contains more than twice as much water as B. How much water could have been poured into B from A? Express your answer by an inequality involving a whole number.

[Solution] If we pour x liters of water from A into B, then

the amount of water in A is $(18 - x)$ liters
the amount of water in B is $(3 + x)$ liters

Also, because A contains more than twice as much water as B, the following inequality can be set up:

$$18 - x > 2(3 + x)$$

We find that

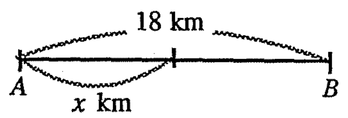
$$x < 4$$

Answer: $x < 4$ liters

Problem 2 Two people, A and B, used 26 sheets of drawing paper. The total number of sheets used was less than or equal to half the number of sheets remaining. If A used 5 sheets of paper, what is the maximum number of sheets that B could have used?

Example 3 We went from point A to point B, 18 km away, first walking 5 km/hour and then 4 km/hour. If the trip took 4 hours or less, what is the shortest possible distance we could have walked at 5 km/hour?

[Approach] If we let the distance we walked at 5 km/hour be x km, the distance we walked at 4 km/hour will be $(18 - x)$ km. Therefore, the following inequality can be set up to express the time of the trip:



$$\frac{x}{5} + \frac{18 - x}{4} \leq 4$$

Problem 3 Find the answer to Example 3 by solving the above inequality.

Problem 4 Could the total time for the trip in Example 3 be "3 hours or less"?

Problem 5 There was a certain amount of water in a container. First we used 4 liters of the water, and then we used half of the remaining water. 5 liters or more water was left. What is the minimum amount of water that could have been in the container initially?

Exercises

1. Solve the following inequalities:

(1) $5 - 2x < 9$

(2) $2x - 4 \leq 3x + 2$

(3) $3x - 4 < 5x - 2$

(4) $5x - 2 > 3x + 5$

(5) $2x - (9x + 4) < 3$

(6) $x - 6 > \frac{5}{2}x$

(7) $0.7x + 0.6 \leq 0.5x + 1$

(8) $0.05x - 1.1 \geq 1.3x - 0.1$

(9) $\frac{x}{3} - 6 \geq \frac{x}{7} - 2$

(10) $\frac{x - 3}{4} - \frac{2x - 1}{3} < x$

2. The number you get by multiplying a certain positive integer by 3 and then adding 2 is greater than the number you get by multiplying the original number by 5 and then subtracting 6. Find all the positive integers for which this statement is true.

3. We want to buy some apples at 70 yen apiece and some pears at 55 yen apiece for a total price less than or equal to 1,300 yen. If we want as many apples as possible, how many of each can we buy?

2 SIMULTANEOUS LINEAR INEQUALITIES

1 Solving Simultaneous Inequalities

An unknown number x multiplied by 2 is greater than x subtracted from 6. Furthermore, the number we get by adding 1 to x is greater than the number we get by subtracting 3 from 2 times x. What is x?

The first condition of this problem can be expressed as the following inequality:

$$2x > 6 - x \qquad (1)$$

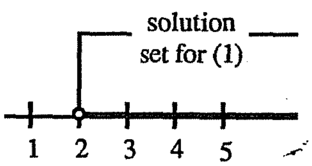
The second condition can be expressed as

$$x + 1 > 2x - 3 \qquad (2)$$

x is a number which satisfies both inequalities (1) and (2).

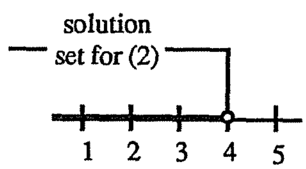
Solving inequality (1), then

$$2x > 6 - x$$
$$2x + x > 6$$
$$3x > 6$$
$$x > 2$$



Solving inequality (2), then

$$x + 1 > 2x - 3$$
$$x - 2x > -3 - 1$$
$$-x > -4$$
$$x < 4$$



If we take the solution set for (1) as A and that for (2) as B, then

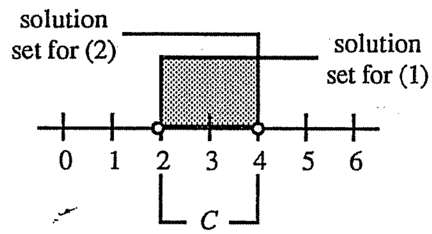
$$A = \{x \mid x > 2\}, B = \{x \mid x < 4\}$$

Because the number x that satisfies both (1) and (2) must be common to both A and B,

it is greater than 2 and smaller than 4.

The solution set C for that number can be expressed as

$$C = \{x \mid 2 < x < 4\}$$



A combination of two separate inequalities, such as (1) and (2) on the preceding page, can be represented in the following way:

$$\begin{cases} 2x > 6 - x & (1) \\ x + 1 > 2x - 3 & (2) \end{cases}$$

Such a combination is said to comprise **simultaneous inequalities.**

Here, both of the inequalities are linear, so the combination can be said to consist of **simultaneous linear inequalities.**

The values of x which satisfy both inequalities are the solutions to the simultaneous inequalities. The solution set for the simultaneous inequalities above is

$$\{x \mid 2 < x < 4\}$$

Solving simultaneous inequalities means finding the solution set. However, we write the answer to simultaneous inequalities in the form "$2 < x < 4$", instead of writing the solution set as $\{x \mid 2 < x < 4\}$.

Problem 1 Can 3.5 be a solution to the simultaneous inequalities above? What about 4?

Problem 2 If x is an integer, what is the solution to the simultaneous inequalities above?

② Solving and Applying Simultaneous Inequalities

Example 1 Solve the following simultaneous inequalities:

$$\begin{cases} 3x - 8 < 1 & (1) \\ 5 - 2x < 9 & (2) \end{cases}$$

[Solution]

Solving inequality (1),

$3x < 1 + 8$

$3x < 9$

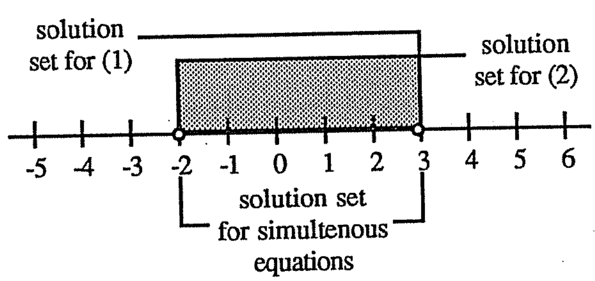
$x < 3$

Solving inequality (2),

$-2x < 9 - 5$

$-2x < 4$

$x > -2$



Thus, $-2 < x < 3$

Answer: $-2 < x < 3$

Problem 1 Solve the following simultaneous inequalities:

(1) $\begin{cases} 6x + 5 > 3x \\ 3x - 2 < 7 \end{cases}$ (2) $\begin{cases} 3x < 4x + 3 \\ 2 - x > 3 \end{cases}$

Example 2 Solve the following simultaneous inequalities:

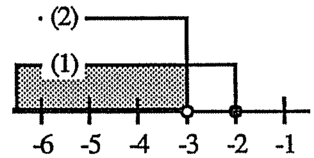
$$\begin{cases} \frac{1}{2}x + 6 \le 5 & (1) \\ 7x > 9x + 6 & (2) \end{cases}$$

[Solution]

Solving inequality (1),	Solving inequality (2),
$x + 12 \le 10$	$7x - 9x > 6$
$x \le 10 - 12$	$-2x > 6$
$x \le -2$	$x < -3$

Thus, $x < -3$

Answer: $x < -3$



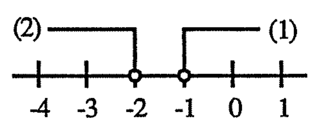
In Example 2, the solution set for inequality (2) is a part of the solution set for inequality (1). Thus, the solution set for these simultaneous inequalities coincides with the solution set for inequality (2).

Example 3 Solve the following simultaneous inequalities:

$$\begin{cases} 2x + 3 > x + 2 & (1) \\ 3x > 4x + 2 & (2) \end{cases}$$

[Solution] If we solve both inequalities (1) and (2), we get

$x > -1, x < -2$



There is no element common to the solution sets for (1) and (2). Thus, there is no solution to these simultaneous inequalities.

Answer: No solution

Problem 2 Solve the following simultaneous inequalities:

(1) $\begin{cases} 2x + 3 \geq 1 \\ x + 3 \leq 16 \end{cases}$ (2) $\begin{cases} 4x - 3 < 2x - 7 \\ 2x + 5 > x + 6 \end{cases}$

(3) $\begin{cases} 3x - 4 \geq 2 \\ 4x - 5 \geq 5x - 7 \end{cases}$ (4) $\begin{cases} 2x + 3 > 3x + 7 \\ \frac{2}{3}x + 1 < -3 \end{cases}$

Example 4 Solve the following simultaneous inequalities:

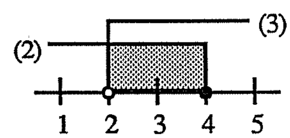
$$2x + 1 \leq x + 5 < 3x + 1 \qquad (1)$$

[Solution] (1) is a combination of the following two inequalities.

$$\begin{cases} 2x + 1 \leq x + 5 & (2) \\ x + 5 < 3x + 1 & (3) \end{cases}$$

If we solve inequalities (2) and (3), we get

$$x \leq 4, x > 2$$



Thus, $2 < x \leq 4$.

Answer: $2 < x \leq 4$

Problem 3 Solve the following simultaneous inequalities:

(1) $5x - 8 \geq 7x > 3x - 28$

(2) $3x - 7 < 2x + 5 \leq 4x + 9$

Example 5 There are 123 notebooks and 23 dozen pencils. If we distribute 3 notebooks to each student in a class, we will have 10 or more notebooks left. If we distribute 8 pencils to each student, we will be short 10 or more pencils. How many students are there in this class?

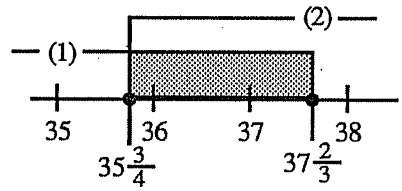
[Solution] If we take x as the number of students in this class, then

based on the number of notebooks: $123 - 3x \geq 10$ (1)
based on the number of pencils: $8x - 12 \times 23 \geq 10$ (2)

If we combine inequalities (1) and (2) and solve them as simultaneous inequalities, we get

from (1): $x \leq 37\frac{2}{3}$

from (2): $x \geq 35\frac{3}{4}$



Thus, $35\frac{3}{4} \leq x \leq 37\frac{2}{3}$ (3)

The number of students in the class must be a natural number, and so the natural numbers which satisfy (3) are 36 and 37.

Answer: 36 or 37

Problem 4 We have 700 manufactured articles to pack in boxes. If we pack 15 articles in each box, enough articles are left over to fill 3 or more boxes. If we pack 20 articles in each box, 8 or more empty boxes are left over. What is the original number of boxes?

Example 6 We want to make a blend of tea which weighs 1,400 g or more and costs 7,200 yen or less. We must mix 600 g of tea A, at 300 yen for 100 g, with tea B, at 450 yen for 100 g. What are the maximum and minimum amounts of tea B that we can add to the blend?

⟦ Approach ⟧ If we let the amount of tea B that we add be x g and express its relation to the total weight and its relation to the total price as inequalities, then we get

based on the weight: $600 + x \geq 1{,}400$ (1)

based on the price: $\frac{300}{100} \times 600 + \frac{450}{100} \times x \leq 7{,}200$ (2)

We can combine (1) and (2) and solve them as simultaneous inequalities.

Problem 5 Solve (1) and (2) in Example 6 as simultaneous inequalities. Then check your answer. Did you find that we can add 800 g or more and 1,200 g or less of tea B?

Problem 6 In Example 6 what would the answer be if we wanted the total price to be 5,200 yen or less?

Exercises

1. Solve the following simultaneous inequalities:

(1) $\begin{cases} 3x - 1 > 2x - 5 \\ 4x - 7 \geq 6x - 11 \end{cases}$

(2) $\begin{cases} 11 \leq 2x - 5 \\ 7 - 4x > 5 - 2x \end{cases}$

(3) $\begin{cases} 5x - 4 \geq 6 \\ \frac{4}{5}(x - 5) \leq 8 \end{cases}$

(4) $\begin{cases} 2x - 5 \leq 6x + 1 \\ 3x - 2 \leq 5x + 7 \end{cases}$

2. It takes a boy 20 minutes or more to get home from school if he walks the first 900 meters at a speed of 60 m/minute and the rest of the distance at 80 m/minute. However, it takes 20 minutes or less if he walks the first 600 meters at a speed of 60 m/minute and the rest of the way at 80 m/minute. Find the maximum and minimum possible distance between the school and the boy's home.

3. We decided to manufacture the same number of articles every day in our shop. 8 days after we started manufacturing them, we were 100 articles short of our goal. Even though we had not reached half of the intended number after 10 days, we surpassed our goal after 21 days. How many articles did we make a day?

Chapter Exercises

A

1. Solve the following inequalities:

(1) $x - 22 > 4$

(2) $25 > x + 8$

(3) $12x \geq -36$

(4) $-\frac{6}{7}x > 6$

(5) $4x + 7 \leq 3$

(6) $28 - 3x < 13$

(7) $5x - 12 > 14 + 9x$

(8) $0.7x - 1.1 \leq 1.8x + 2.2$

(9) $3 - \frac{1}{6}x \geq -\frac{1}{2}x + 8$

(10) $5 - (2x - 3) < 3x + 18$

2. Solve the following simultaneous inequalities:

(1) $\begin{cases} 2x + 5 \leq 9 \\ 3x + 1 > 10 \end{cases}$

(2) $\begin{cases} 3x + 4 \geq 4x + 7 \\ 4x + 5 \geq 2x - 1 \end{cases}$

(3) $\begin{cases} 4x - 1 > 2x - 7 \\ 2(x - 3) > 6x - 2 \end{cases}$

(4) $5x - 9 \leq 2x < 7x + 10$

3. A family's telephone bill is 2,500 yen or more every month. The phone bill consists of a basic charge and a charge for local calls. The basic charge is 1,800 yen a month, and each local call up to 3 minutes long costs 10 yen. If this family made only local calls of 3 minutes or less, what is the minimum number of local calls that the family makes every month?

4. When we add 8 to an unknown number and multiply the sum by 3, the result is greater than 36. When we subtract the unknown number from 7 and multiply the difference by 2, the result is greater than 2. Find the range of the unknown number.

5. A freight train 22 cars long includes some cars carrying 15 tons and some carrying 20 tons. The total weight of the freight is 360 tons or more and 400 tons or less. Find the minimum and maximum number of cars carrying 20 tons.

B

1. Solve the following inequalities:

(1) $\dfrac{3x-1}{2} - x > 4$ (2) $\dfrac{x}{2} - \dfrac{x-5}{3} > \dfrac{5}{6}$

(3) $\dfrac{x-5}{2} - \dfrac{2x+1}{3} < 1$ (4) $\dfrac{3x-1}{4} > \dfrac{4x-2}{3} + 1$

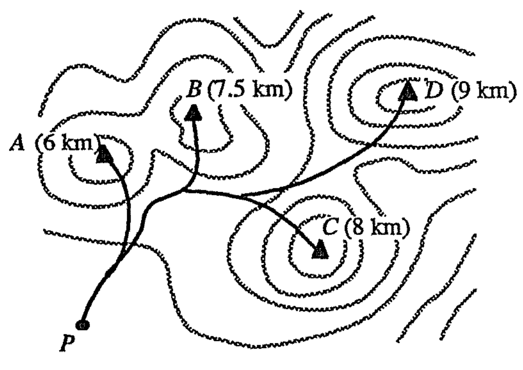
2. Solve the following simultaneous inequalities:

(1) $\begin{cases} 2x \geq x - 1 \\ 5(x-2) \leq 0 \end{cases}$ (2) $\begin{cases} 5x + 2 > 7x - 6 \\ \dfrac{x+6}{3} \geq \dfrac{x-1}{2} - x \end{cases}$

(3) $3x - 6 < 2x + 1 < \dfrac{5x-1}{2}$ (4) $\dfrac{x-2}{3} \geq x \geq \dfrac{3x-1}{4}$

(5) $\begin{cases} 8x - 5 \geq 3(4x-1) \\ 2x - \dfrac{x-2}{3} < 4 \end{cases}$ (6) $\begin{cases} \dfrac{x}{4} - \dfrac{1-x}{6} \geq -1 \\ 3(1-x) \geq -2x - 1 \end{cases}$

3. We plan to climb one of the mountains A, B, C, and D on Sunday. The numbers on the map to the right show the distances from P to the top of each mountain. We plan to leave P at 9:00 in the morning, climb the most distant mountain possible, rest for an hour at the peak, and return to P by 3:00 in the afternoon. Which mountain can we climb, if we hike at a speed of 3 km/hour on the way to the top and 4 km/hour on the way back?



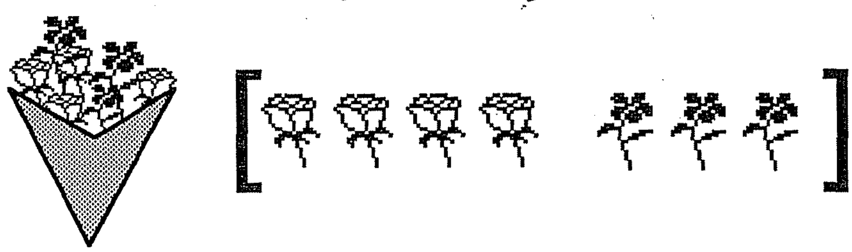
4. We want to mix 400 g of an alloy that contains 60% silver with some amount of a 75% silver alloy to produce 600 g or more of an alloy that contains 420 g or less of silver. What are the maximum and minimum amounts of the 75% silver alloy that we may use?

CHAPTER 3

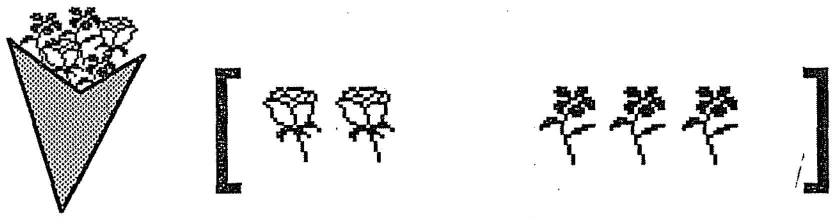
SIMULTANEOUS EQUATIONS

A certain flower shop sells roses and lilies in the following bunches:

1,900 yen for 4 roses and 3 lilies



1,400 yen for 2 roses and 3 lilies



How much would one rose cost? One lily?

1 SIMULTANEOUS EQUATIONS

1 Linear Equations in Two Variables and the Solutions to Simultaneous Equations

An amusement park has two kinds of bicycles for rent: tandem bikes and ordinary bikes. A group of 9 junior high school students want to rent one or more of both kinds so that everyone can ride. How many tandem bikes and ordinary bikes should they rent?

Problem 1 In the above problem, assuming that we rent x tandem bikes and y ordinary bikes, express the relation between x and y in an equation.

In order to solve the original problem, we must find the paired values of natural numbers x, y which satisfy the equation we formulated in Problem 1:

$$2x + y = 9 \tag{1}$$

A linear equation that contains two letters, such as (1), is called a **linear equation in two variables.**

Problem 2 The table to the right represents the paired values of x, y which satisfy (1). Fill in the blanks in the diagram.

x	1	2		4
y	7		3	

If we represent the paired values of x, y from Problem 2 in the form of (x, y), we find that the following four pairs of values satisfy the conditions of the original problem:

$$(1, 7),\ (2, 5),\ (3, 3),\ (4, 1)$$

Now let's consider a new problem which adds to the original problem the stipulation that "they want to rent a total of 7 bikes."

Assuming that they rent x tandem bikes and y ordinary bikes, we can express this stipulation as

$$x + y = 7 \qquad (2)$$

In order to solve the new problem, we must examine the paired values of x, y which satisfy (1) and also determine which of them satisfy (2) as well.

Problem 3 Which of the paired values of x, y below satisfy (1), and which satisfy (2)?

$$(1, 7),\ (2, 5),\ (3, 3),\ (4, 1)$$

As we saw in Problem 3, the paired values of x, y we were looking for are (2, 5). Therefore, the students can rent 2 tandem bikes and 5 ordinary bikes.

Two or more equations like the combination of (1) and (2) given here are called **simultaneous equations.**

$$\begin{cases} 2x + y = 9 \\ x + y = 7 \end{cases}$$

The paired values of the variables that satisfy both equations are called the **solutions** to the simultaneous equations; moreover, finding the solutions is referred to as solving the simultaneous equations. So $x = 2$ and $y = 5$ are the solutions to the above simultaneous equations.

The simultaneous equations above are a combination of linear equations in two variables. Such simultaneous equations are called **simultaneous linear equations in two variables.**

2 Solving Simultaneous Linear Equations in Two Variables

Let's try solving simultaneous equations by calculating. To do this, we must first derive an equation which contains only one variable from the simultaneous equations.

The Addition-Subtraction Method

Example 1 Solve the following simultaneous equations:

$$\begin{cases} x + y = 13 & (1) \\ 2x - y = 5 & (2) \end{cases}$$

[Approach] If we add the left sides of the equations, we get one equation without y. The sum of the left sides should equal the sum of the right sides.

$$\begin{array}{rcl} A & = & B \\ +) \quad C & = & D \\ \hline A + C & = & B + D \end{array}$$

[Solution] If we add both left sides together and both right sides together,

$$\begin{array}{rcl} x + y & = & 13 \\ +) \quad 2x - y & = & 5 \\ \hline 3x & = & 18 \end{array} \qquad (3)$$

$$x = 6 \qquad (4)$$

Substituting (4) into (1),

$$6 + y = 13$$
$$y = 7$$

Answer: $x = 6$, $y = 7$

[Check] If we substitute these values for x and y back into the original simultaneous equations,

in (1): left side $= 6 + 7 = 13$; right side $= 13$
in (2): left side $= 2 \times 6 - 7 = 5$; right side $= 5$

Note: Henceforth, checks will not be given in this textbook.

Problem 1 Find the value of y in Example 1 by substituting (4) into (2), and compare the results.

Equation (3) includes only the variable x, not y.

Deriving an equation that does not include y from an equation that does include y is called **eliminating** y.

Example 2 Solve the following simultaneous equations:

$$\begin{cases} 5x + 2y = 9 & (1) \\ 5x - 3y = -1 & (2) \end{cases}$$

[Approach] If we find the difference of the left sides of the equations, x will be eliminated.

$$\begin{array}{rl} & A = B \\ -) & C = D \\ \hline & A - C = B - D \end{array}$$

[Solution] Subtracting both sides of (2) from both sides of (1),

$$\begin{array}{rrcl} & 5x + 2y & = & 9 \\ -) & 5x - 3y & = & -1 \\ \hline & 5y & = & 10 \end{array}$$

$$y = 2 \qquad (3)$$

Substituting (3) into (1),

$$5x + 2 \times 2 = 9$$

$$5x = 5$$

$$x = 1$$

Answer: $x = 1, y = 2$

The method of solving simultaneous equations adopted in Examples 1 and 2 – by adding and subtracting both sides of the equations to eliminate one variable – is called the **addition-subtraction method.**

Problem 2 Solve the following simultaneous equations by the addition - subtraction method:

(1) $\begin{cases} x + y = -3 \\ x - y = 7 \end{cases}$ (2) $\begin{cases} 7x - 5y = 3 \\ 7x - 6y = 12 \end{cases}$

(3) $\begin{cases} 3x + 2y = 23 \\ 5x + 2y = 29 \end{cases}$ (4) $\begin{cases} 2x + y = 2 \\ -y + 5x = -9 \end{cases}$

Example 3 Solve the following simultaneous equations:

$$\begin{cases} 7x - 2y = 29 & (1) \\ -2x + y = -10 & (2) \end{cases}$$

[Approach] If we just add (1) and (2), we cannot eliminate either x or y. Therefore, we must change the coefficients of y in (1) and (2) into the same number, and then add the equations.

[Solution]

(1) $7x - 2y = 29$

(2) x 2 $+)\ -4x + 2y = -20$

$3x = 9$

$x = 3$ (3)

Substituting (3) into (2),

$-2 \times 3 + y = -10$

$y = -4$

Answer: $x = 3, y = -4$

Problem 3 Solve the following simultaneous equations:

(1) $\begin{cases} 2x + 3y = 5 \\ x + 2y = 4 \end{cases}$ (2) $\begin{cases} 2x - y = 4 \\ 5x + 3y = -1 \end{cases}$

(3) $\begin{cases} 5x - 3y = 3 \\ 10x + 5y = 6 \end{cases}$ (4) $\begin{cases} 6x - 7y = 12 \\ 2y - 3x = 3 \end{cases}$

Example 4 Solve the following simultaneous equations:

$$\begin{cases} 3x - 4y = -15 & (1) \\ 2x + 3y = 7 & (2) \end{cases}$$

[Solution]

(1) × 2 $\quad 6x - 8y = -30$

(2) × 3 $\quad -)\ 6x + 9y = 21$

$$-17y = -51$$

$$y = 3 \qquad (3)$$

Substituting (3) into (1),

$$3x - 4 \times 3 = -15$$

$$3x = -3$$

$$x = -1$$

Answer: $x = -1, y = 3$

Problem 4 Solve Example 4 by first eliminating y.

Problem 5 Solve the following simultaneous equations:

(1) $\begin{cases} 4x + 7y = -13 \\ 5x + 2y = 4 \end{cases}$ (2) $\begin{cases} 4m - 5n = 21 \\ 3m - 2n = 21 \end{cases}$

(3) $\begin{cases} 5a + 3b = 2 \\ 9a - 2b = 11 \end{cases}$ (4) $\begin{cases} -3x + 7y = -1 \\ 5x - 4y = -6 \end{cases}$

The Substitution Method

There is another way to eliminate a variable, as illustrated in the following example.

Example 5 Solve the following simultaneous equations:

$$\begin{cases} y = 2x - 3 & (1) \\ 5x - 4y = 6 & (2) \end{cases}$$

[Approach] If we substitute for y in (2) with $2x - 3$, which is equal to y in (1), then the y in (2) will be eliminated.

$y = (2x - 3)$ ......(1)

↓

$5x - 4(y) = 6$ ......(2)

[Solution] If we substitute (1) into (2), we get

$$5x - 4(2x - 3) = 6$$

Rearranging the terms,

$$-3x = -6$$

$$x = 2 \qquad (3)$$

Substituting (3) into (1),

$$y = 2 \times 2 - 3 = 1$$

Answer: $x = 2, y = 1$

In Example 5 we eliminated y by substituting (1) into (2). This method of solving simultaneous equations is called the **substitution method.**

Problem 6 Solve the following simultaneous equations by the substitution method:

(1) $\begin{cases} y = 2x \\ x + y = 6 \end{cases}$ (2) $\begin{cases} x + 4y = -15 \\ x = 4y + 1 \end{cases}$

(3) $\begin{cases} y = 11 - 2x \\ 7x - 9y = 1 \end{cases}$ (4) $\begin{cases} 4x + 3y = 3 \\ 3y = -5x \end{cases}$

There are two ways of solving the simultaneous equations: by the addition-subtraction method and by the substitution method. But the two methods are essentially the same – both are used to eliminate one letter in solving simultaneous equations. Depending on the situation, either method may be used – whichever way is easier to calculate.

Problem 7 Solve the following simultaneous equations by an appropriate method:

(1) $\begin{cases} 5x + 2y = 10 \\ 3x + 2y = 6 \end{cases}$ (2) $\begin{cases} 7x - 4y = -3 \\ 7x + y = -8 \end{cases}$

(3) $\begin{cases} y = x + 1 \\ y = -2x + 13 \end{cases}$ (4) $\begin{cases} 3x - 2y = -1 \\ 9x - 5y = 5 \end{cases}$

(5) $\begin{cases} y = 3x - 1 \\ x - 2y = 12 \end{cases}$ (6) $\begin{cases} 3x + 2y = -19 \\ 2x + 3y = -6 \end{cases}$

Drills

1. Solve the following simultaneous equations:

(1) $\begin{cases} 2x + y = -5 \\ x + 3y = 5 \end{cases}$ (2) $\begin{cases} y = 2x - 9 \\ y = 1 - 3x \end{cases}$

(3) $\begin{cases} 7m - 5n = 17 \\ 8m + 3n = 63 \end{cases}$ (4) $\begin{cases} 9x + 10y = 12 \\ 2x - 5y = -6 \end{cases}$

③ Various Types of Simultaneous Equations

There are various types of simultaneous linear equations in two variables.

Example 1 Solve the following simultaneous equations:

$$4x + y = 3x - y = 7 \qquad (1)$$

[Solution] We can rewrite (1) in this way.

$$\begin{cases} 4x + y = 7 \\ 3x - y = 7 \end{cases}$$

Solving these equations,

$$x = 2, \quad y = -1$$

Answer: $x = 2$, $y = -1$

We can solve simultaneous equations of the form $A = B = C$, as illustrated in Example 1, in any of the following combinations:

$$\begin{cases} A = B \\ A = C \end{cases} \qquad \begin{cases} A = B \\ B = C \end{cases} \qquad \begin{cases} A = C \\ B = C \end{cases}$$

Problem 1 Solve the following simultaneous equations:

(1) $2x + y = x + 3y = 5$

(2) $4x + 9y = -10x - 12y = -1$

Example 2 Solve the following simultaneous equations:

$$\begin{cases} 3(x - y) + 2y = 11 & (1) \\ 5x - 2(3x - y) = -7 & (2) \end{cases}$$

[Approach] If we remove the parentheses and rearrange the terms, these equations reduce to

$$\begin{cases} 3x - y = 11 & (3) \\ -x + 2y = -7 & (4) \end{cases}$$

We can solve them now easily.

Problem 2 Solve the simultaneous equations in Example 2.

When the coefficients in simultaneous equations contain fractions and decimals, it is convenient first to transform all the coefficients into integers and then solve the equations.

Example 3 Solve the following simultaneous equations:

$$\begin{cases} 0.3x + 0.4y = -3 & (1) \\ \frac{1}{2}x - \frac{1}{3}y = 1 & (2) \end{cases}$$

[Approach] If we transform all the coefficients in (1) and (2) into integers, we get

$$\begin{cases} 3x + 4y = -30 & (3) \\ 3x - 2y = 6 & (4) \end{cases}$$

Problem 3 Solve the simultaneous equations in Example 3.

Problem 4 Solve the following simultaneous equations:

(1) $\begin{cases} 5x + 3y = -1 \\ 3x - 4(x + y) = 7 \end{cases}$ (2) $\begin{cases} 0.2x - 1.4y = 5 \\ \frac{1}{4}x + \frac{2}{3}y = -1 \end{cases}$

Drills

1. Solve the following simultaneous equations:

(1) $\begin{cases} 3(x - 2y) + 5y = 2 \\ 4x - 3(2x - y) = 8 \end{cases}$ (2) $\begin{cases} 0.2x - 0.3y = 2.2 \\ -2x + y = -6 \end{cases}$

(3) $\begin{cases} 2x - \frac{y}{3} = 3 \\ \frac{x}{2} + y = 4 \end{cases}$ (4) $\begin{cases} \frac{x}{4} - \frac{y}{3} = 4 \\ -\frac{x}{3} + \frac{y}{5} = -20 \end{cases}$

4 Applying Simultaneous Linear Equations in Two Variables

We bought some oranges at 40 yen apiece and some apples at 90 yen apiece, a total of 12 oranges and apples for 880 yen. How many of each fruit did we buy?

In this problem, we are given the following information:

(the number of oranges) + (the number of apples) = 12

40 x (the number of oranges) + 90 x (the number of apples) = 880

Assuming that we bought x oranges and y apples, we have the following two equations:

$$x + y = 12 \quad (1)$$

$$40x + 90y = 880 \quad (2)$$

Problem 1 Solve equations (1) and (2) above as simultaneous equations, and find the number of oranges and the number of apples.

This probem can be solved using only one variable.

If the number of oranges is x, then the number of apples will be $(12 - x)$. Thus,

$$40x + 90(12 - x) = 880$$

Solving this equation, $x = 4$

The number of apples is: $12 - 4 = 8$

However, it is often easier to construct equations using two variables.

Problem 2 We bought some 20-yen stamps and some 50-yen stamps, a total of 12 stamps for 450 yen. How many stamps of each denomination did we buy?

Example 1 There are two kinds of metal bars, type A and type B. The weight and price per meter for A and B are given in the table to the right. If we combine certain quantities of A and B, the total weight is 12 kg and the total price is 1,100 yen. How many meters of A and how many meters of B do we have?

	A	B
weight (kg)	2	3
price (yen)	200	250

[Solution] Taking A to be x m long and B y m long,

$$\begin{cases} 2x + 3y = 12 & (1) \\ 200x + 250y = 1100 & (2) \end{cases}$$

Dividing both sides of (2) by 50,

$$4x + 5y = 22 \qquad (3)$$

$$\begin{array}{lrcl} (1) \times 2 & 4x + 6y & = & 24 \\ (3) & -)\ 4x + 5y & = & 22 \\ \hline & y & = & 2 \end{array} \qquad (4)$$

Substituting (4) into (1),

$$2x + 6 = 12$$
$$x = 3$$

Answer: A is 3 m long, B is 2 m long

Problem 3 4 pencils and 3 notebooks cost 520 yen, and 8 pencils and 5 notebooks cost 920 yen. What are the prices of one pencil and one notebook?

Problem 4 There are two kinds of medicine, A and B. The combined weight of 5 liters of A and 2 liters of B is 9 kg, and the combined weight of 10 liters of A and 6 liters of B is 21 kg. Find the weight per liter of each medicine.

Example 2 Driving from town A to town B 90 km away, a woman took the expressway part of the way. It took her 1 hour and 21 minutes to get to town B. She drove 80 km/hour on the expressway and 50 km/hour on local roads. Find the distance that she drove on the expressway and on local roads.

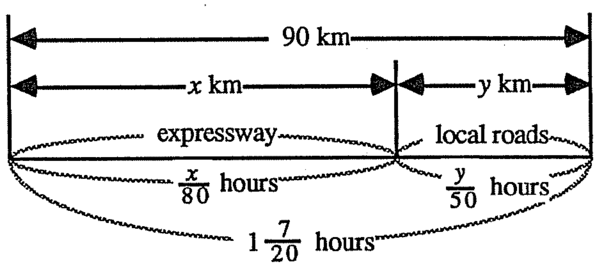
⟦ Approach ⟧ Assuming that she drove x km on the expressway and y km on local roads, we can express the relation between the two distances as

$$x + y = 90 \qquad (1)$$

and the relation between the times driven as

$$\frac{x}{80} + \frac{y}{50} = 1\frac{7}{20} \qquad (2)$$

Then, we can solve these equations.



Problem 5 Solve (1) and (2) above as simultaneous equations, and check whether the distance on the expressway is 60 km and the distance on local roads is 30 km.

Problem 6 Solve the problem in Example 2 by letting the time on the expressway be x hours and the time on local roads be y hours.

As you can see, when we construct equations to go with a problem, the equations may be very easy, depending on what aspects of the problem we express in variables.

Problem 7 A man went from town A to town B 90 km away by crossing a mountain pass. It took him 2 hours and 20 minutes. He walked from town A to the top of the pass at 3 km/hour, and from the pass to town B at 5 km/hour. Find the distances from town A to the top of the pass and from the pass to town B.

Example 3 We want to make 300 g of a salt solution containing 6% salt from an 8% salt solution and a 5% salt solution. How many grams of each solution should we mix together?

[Approach] Suppose we use x g of the 8% solution and y g of the 5% solution. Then the equation for the weight of the mixture will be

$$x + y = 300 \qquad (1)$$

salt	8%	5%	6%
solution (g)	x	y	300
salt (g)	$\frac{8}{100}x$	$\frac{5}{100}y$	$300 \times \frac{6}{100}$

We can determine the amount of salt contained in each solution, and we find:

x g of 8% solution: $\frac{8}{100}x$ g

y g of 5% solution: $\frac{5}{100}y$ g

300 g of 6% solution: $300 \times \frac{6}{100} = 18$ (g)

Thus,

$$\frac{8}{100}x + \frac{5}{100}y = 18 \qquad (2)$$

Rearranging (1) and (2)

$$x + y = 300 \qquad (3)$$

$$8x + 5y = 1800 \qquad (4)$$

Problem 8 Solve equations (3) and (4) in Example 3 as simultaneous equations.

Problem 9 One alloy contains 91% silver, and another contains 86% silver. How many grams of each alloy should we mix to make 100 g of an alloy containing 90% silver?

Exercises

1. Solve the following simultaneous equations:

(1) $\begin{cases} x = y + 2 \\ 3x + y = 14 \end{cases}$

(2) $\begin{cases} 3a - b = 6 \\ 6a + 3b = -8 \end{cases}$

(3) $\begin{cases} -7l + 8m = 38 \\ l + 3m = 7 \end{cases}$

(4) $\begin{cases} 2x - 5(x + y) = 6 \\ 5x - 3(x - y) = 4 \end{cases}$

(5) $\begin{cases} \dfrac{x + y}{2} = 1 \\ x - y = 4 \end{cases}$

(6) $\begin{cases} x + 2y = 4 \\ y - \dfrac{1 - x}{3} = \dfrac{1}{2} \end{cases}$

2. The total price of 500 g of coarse tea and 300 g of green tea is 1,650 yen. The total price of 700 g of coarse tea and 500 g of green tea is 2,550 yen. What is the price of 100 g of each tea?

3. We went to the post office to buy some 10-yen and 15-yen stamps. We expected to receive 5 yen in change from 100 yen. But we were actually 5 yen short because we asked for the wrong number of 10-yen and 15-yen stamps. How many 10-yen stamps did we mean to buy?

4. The student population at a certain school was 850 last year. But this year it increased by 30, because the number of male students increased 4% and the number of female students 3%. How many male and female students are there this year?

Chapter Exercises

A

1. Solve the following simultaneous equations:

(1) $\begin{cases} 3x + 2y = 5 \\ x - 2y = 7 \end{cases}$ (2) $\begin{cases} 6a - b = 1 \\ 3a - 2b = -7 \end{cases}$

(3) $\begin{cases} 4x - 7y = -6 \\ 6x + 2y = -9 \end{cases}$ (4) $\begin{cases} y = 1 + x \\ 2x + 3y = 5 \end{cases}$

(5) $\begin{cases} 2x - 5y = 20 \\ -3(x - y) + y = -2 \end{cases}$ (6) $\begin{cases} x - y = 10 \\ 4x - 1 = -\frac{y}{3} \end{cases}$

(7) $2x + y = 4x + 5y + 2 = x - 3y - 7$

2. If $x + y = 6$ and $x - y = 4$, what is the value of $x^2 - xy + y^2$?

3. Determine the values of a and b in the following simultaneous equations for which the solutions are $x = 2$ and $y = -1$.

$$\begin{cases} ax + by = 4 \\ bx + ay = 1 \end{cases}$$

4. In an equation of the form $y = ax + b$, $y = 6$ for $x = 1$, and $y = 16$ for $x = 3$. Find the value of a and b.

5. In a class of 42 students, $\frac{1}{6}$ of the male students and $\frac{1}{9}$ of the female students wear glasses. The total number of students who wear glasses is $\frac{1}{7}$ of the total number of students in the class. How many male and how many female students are there in this class?

6. We have two different salt solutions, A and B. If we mix 100 g of each solution, we get a solution containing 10% salt. If we mix 400 g of A and 600 g of B, we get a solution containing 9% salt. What is the percentage of salt in each solution?

B

1. Solve the following simultaneous equations:

(1) $\begin{cases} \frac{1}{5}(x + y) = 1 + x \\ \frac{3}{4}y = \frac{5}{4} - \frac{x}{3} \end{cases}$ (2) $\begin{cases} y = -3(x + 1) \\ 1.5x + 0.2y = 1.2 \end{cases}$

2. A two-digit natural number is 7 times the sum of its two digits. If we reverse the two digits, we get a number that is 27 less than the original number. Find the original natural number.

3. 150 people signed up for a trip to a zoo, so we decided to charter 3 buses. As a result, the cost for each person, including admission to the zoo and bus fare, was 1,120 yen. However, another 30 people later signed up, so we added another bus. Consequently, the cost for each person increased another 120 yen. Find the price of admission and the cost of chartering one bus.

4. One day, we counted the passengers who got on and off trains at a station. The total number of passengers in the morning was 1,035. However, the number of passengers who got on trains in the afternoon decreased by $\frac{1}{3}$ compared to the morning, and the number of passengers who got off trains decreased by 205. As a result, the afternoon total was exactly the same as the morning total. Determine the number of passengers who got on and off trains in the morning.

5. Simultaneous equations (1) and (2) have the same solutions. Find the value of a and b.

(1) $\begin{cases} 4x + 7y = 29 \\ 5x - 2y = 4 \end{cases}$ (2) $\begin{cases} ax + by = 4 \\ 3ax - 4by = 26 \end{cases}$

Tracing the Roots of Mathematics

The History of Equations

An Egyptian scroll dating from the second millenium B.C. contains a problem like this:

If we take a certain number and add $\frac{2}{3}$, $\frac{1}{2}$, and $\frac{1}{7}$ of the number to it, the sum is 37. What is the number?

If we write an equation for this problem as we do today, we get

$$x + \frac{2}{3}x + \frac{1}{2}x + \frac{1}{7}x = 37$$

About the third century, a Greek mathematician named Diophantus solved various equations using symbols. But he avoided problems whose answers were negative numbers. On the other hand, in the twelfth century, Bhaskara in India solved a variety of equations, including one with a negative answer.

In an ancient Chinese mathematics textbook, Chiuchang-Suanshu (The Nine Books of Arithmetic), which is said to have been written 2,000 years ago, we find a discussion of how to solve simultaneous linear equations in two variables by applying the addition-subtraction methods to problems such as the following:

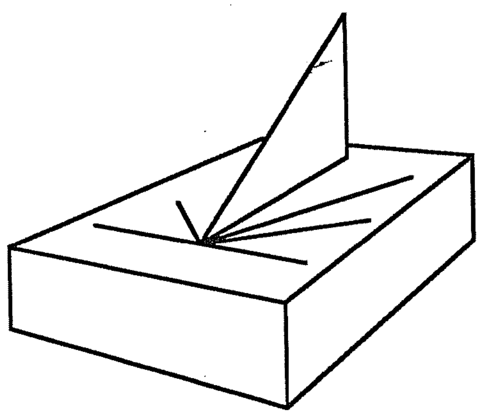
If 5 cattle and 2 sheep cost 10 units of gold, and 2 cattle and 5 sheep cost 8 units of gold, how much does one cow cost, and how much does one sheep cost?

In the beginning of the Edo period [1600-1867], Tienyuan-Shu was introduced to Japan. Tienyuan-Shu was developed in China in the thirteenth century. It is a method of solving equations using calculating rods called sangi. Japanese mathematicians such as Seki Takakazu improved this method, devised symbols, and developed a method called tenzan-zyutsu for solving equations using only written calculations.

Thanks to tenzan-zyutsu, mathematics in Japan, despite the government's isolationist policy, developed on its own and made advances as remarkable as in the West.

CHAPTER 4

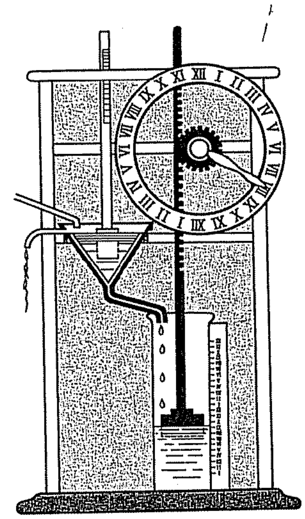
LINEAR FUNCTIONS



People were already interested in time very early in history. It is said that the ancient Babylonians measured time in hours and minutes as early as 5,000 years ago.

How did early civilizations measure time?

The most common instruments made use of things that change at a regular rate as time passes; the sundial, the water clock, and the fire clock are especially well-known. The illustration at the right demonstrates the principle of the water clock, which makes use of the fact that water trickles through a tiny hole at a regular rate.



In this chapter, we will learn about functions that express relations between uniformly changing quantities, like the relation between the amount of water and the time shown by a water clock.

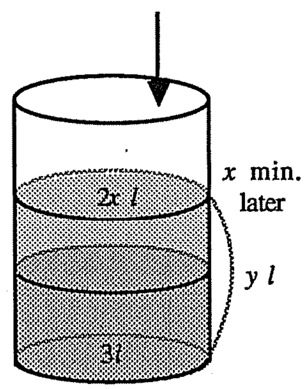
1 LINEAR FUNCTIONS

1 Linear Functions

A tank contains 3 liters of water. Then we let water into the tank for 5 minutes at a rate of 2 liters a minute.

If the total amount of water in the tank is y liters after adding water for x minutes, then y is a function of x, as indicated by the following expression:

$$y = 2x + 3 \qquad (1)$$



Problem 1 For function (1), find the values of y that correspond to various values of x, and fill in the blanks in the following table with the values you find.

x	0	1	2	3	4	5
y						

Problem 2 In function (1), the value of x ranges from $0 < x < 5$. Express the range of values for y as an inequality.

Any function, such as function (1), which gives a linear expression of x is called a **linear function**. For example, the functions

$$y = 3x - 5, \qquad y = -x + 3, \qquad y = \frac{2}{3}x$$

are linear functions because y is given as a linear expression of x.

Problem 3 It is known that the speed of sound in the air is 331 m/second at a temperature of 0°C, and it increases by 0.6 m/second for each 1°C increase in temperature. Assuming that the speed of sound is y m/second at a temperature of x °C, express y in terms of x.

A linear function is given by the general formula:

$$y = ax + b, \text{ where } a \text{ and } b \text{ are constants, and } a \neq 0$$

If $b = 0$, the formula becomes $y = ax$, and y is proportional to x. So a function where y is proportional to x is a special case in which the constant b in the general linear function is equal to 0.

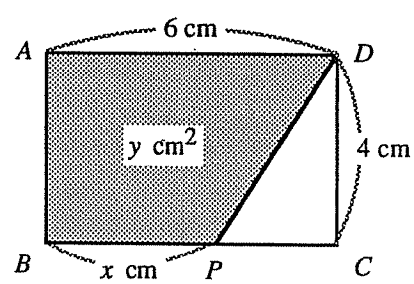
The general linear function is expressed as the sum of ax, the proportional to the variable, and b, the constant part.

y	=	ax	+	b
		(the part proportional to x)		(constant part)

Problem 4 Which part of function (1) is proportional to x ?

Problem 5 We have a car which uses 0.1 liters of gasoline per kilometer. We set out with 35 liters of gasoline in the tank. If we take the amount of gasoline after x km of driving to be y liters, express y in terms of x. Then find the amount of gasoline left after driving 50 km.

Problem 6 In the rectangle at the right, point P moves from B to C along side BC. Assuming that BP is x cm in length and the area of polygon $ABPD$ is y cm^2, express y in terms of x. Then, find the range of the values of x and y.



2 Change in the Values of Linear Functions

Let's examine how the values of the variables in the following linear function can change.

$$y = 2x + 3 \qquad (1)$$

1 1 1 1 1 1 1 1 1 1

x	....	-5	-4	-3	-2	-1	0	1	2	3	4	5	....
$2x$	....	-10	-8	-6	-4	-2	0	2	4	6	8	10	....
y	....	-7	-5	-3	-1	1	3	5	7	9	11	13	....

Problem 1 If the value of x in function (1) increases by increments of 1, by what increments does the value of y increase? If the value of x increases by increments of 3, by what increments does the value of y increase?

Example 1 Let's examine how the value of y in function (1) changes as the value of x increases from 1 to 6.

The increase in x is

$$6 - 1 = 5$$

and the increase in y is

$(2 \times 6 + 3) - (2 \times 1 + 3)$
$= 10$

Thus, the increase in y is 2 times greater than the increase in x.

$$\frac{(\text{increase in } y)}{(\text{increase in } x)} = 2$$

5

x	•••	1	•••	6	•••
y	•••	5	•••	15	•••

10

Problem 2 In function (2), find $\frac{(\text{increase in } y)}{(\text{increase in } x)}$ when the value of x increases from 4 to 7.

Problem 3 Given the linear function $y = 3x - 2$, find $\frac{\text{(increase in } y)}{\text{(increase in } x)}$ when the value of x increases:

(1) from 1 to 4 (2) from -6 to -2.

In the general linear function $y = ax + b$,

$$\frac{\text{(increase in } y)}{\text{(increase in } x)} = 2$$

is constant and equal to a. This constant value is called the **rate of change** of a linear function.

$$\frac{\text{(increase in } y)}{\text{(increase in } x)} = a$$

$$y = \underset{\substack{\downarrow \\ \text{rate of} \\ \text{change}}}{\underline{a}}\,x + \underset{\substack{\downarrow \\ \text{value of } y \\ \text{for } x = 0}}{\underline{b}}$$

From this formula,

$$\text{(increase in } y) = a \times \text{(increase in } x)$$

Thus, the increase in y is proportional to the increase in x. The constant a is the increase in y when x increases by 1.

Problem 4 For each of the following linear functions, find the increase in y when x increases by 4.

(1) $y = \frac{1}{2}x - 1$ (2) $y = -3x + 5$

Problem 5 The following table gives several values for x and y in a certain linear function. Find the numbers that go in the blanks, and express y in terms of x.

x	-4	-2	0	2	4	6
y		-7		-1	2	

③ Graphs of Linear Functions

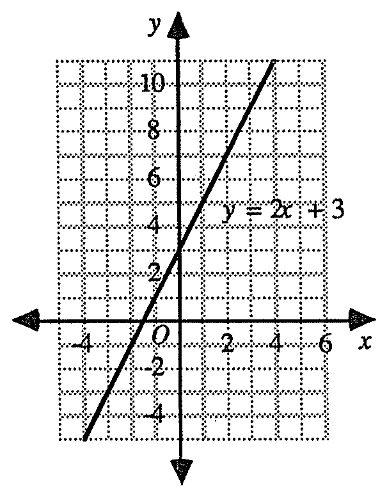
Let's make a table of values for the linear function

$$y = 2x + 3 \qquad (1)$$

and then draw its graph.

x	.....	-4	-3	-2	-1	0	1	2	3	4	.....
y	.....	-5	-3	-1	1	3	5	7	9	11	.....

Problem 1 Draw coordinate axes on a sheet of graph paper, and plot the points whose coordinates are given by paired values of x, y.



If we make the table more detailed and plot more points on the graph, it takes on the form of a straight line, as shown in the diagram to the right.

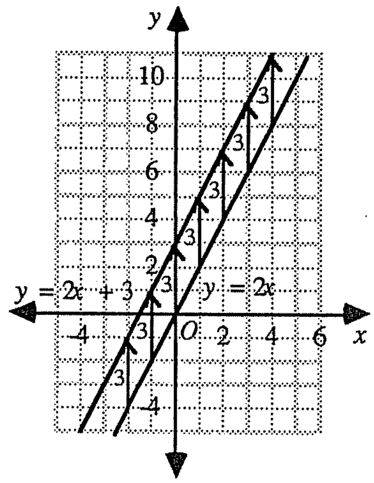
This straight line is the set of all the points whose coordinates are given by the paired values x, y which satisfy the relation in (1).

Problem 2 Which of the following points are on the graph of (1)? Answer by substituting x-coordinates and y-coordinates into equation (1).

A (5,13) B (-6, -10)

C (8, 20) D (-9, -15)

Problem 3 In the water tank example on page 71, express the relation $y = 2x + 3$ by a graph, taking into account the range of values of x, y.



Next, compare the graph of (1) to the graph of the linear function,

$$y = 2x \quad (2)$$

When we compare equations (1) and (2), for each value of x the corresponding value of y in (1) is 3 more than the corresponding value of y in (2). Therefore, the points on the graph of (1) make up a line equivalent to moving the points of (2) up exactly 3 squares.

Problem 4 Graph the following pairs of linear functions, using the same coordinate axes for each pair. State the relations shown by the graphs.

(1) (a) $y = -2x$
(b) $y = -2x + 3$

(2) (a) $y = \frac{1}{2}x$
(b) $y = \frac{1}{2}x + 4$

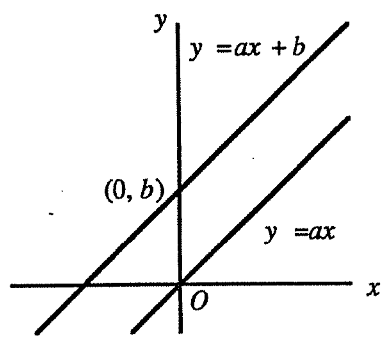
Relations in Proportional Graphs

The graph of a linear function $y = ax + b$ is a straight line formed by moving the graph of $y = ax$ a distance of b along the positive direction of the y-axis, while it remains parallel to the original line.

Note that "moving a distance of b along the positive direction of the y-axis" means that if b is negative, for example $b = -5$, the graph of $y = ax$ moves a distance of 5 along the negative direction of the y-axis.

The constant b of the linear function $y = ax + b$ is the value of y for $x = 0$, and it is the y-coordinate of the point $(0, b)$ at which the graph intersects the y-axis. This constant b is called the **intercept** of the graph of the linear function $y = ax + b$.

The graph of $y = ax + b$ is a straight line passing through the point $(0, b)$ parallel to the graph of $y = ax$.



Problem 5 Give the intercept for the graph of the linear function in Problem 4.

Slope of a Graph

Let's consider the way the graph of the linear function $y = 2x + 3$ inclines.

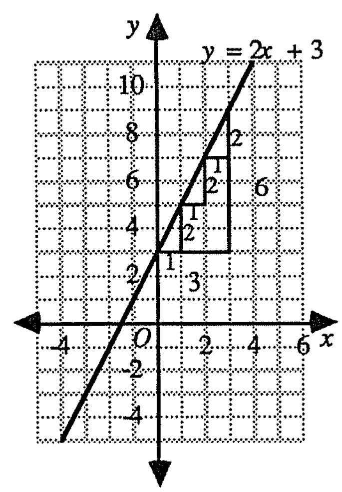
Problem 6 Give the rate of change for the above function.

In the linear function $y = 2x + 3$,

$$\frac{(\text{increase in } y)}{(\text{increase in } x)} = 2.$$

Therefore, when the line of the graph moves 1 square to the right, it moves up 2 squares. In general, the line moves up $2h$ when it moves h to the right.

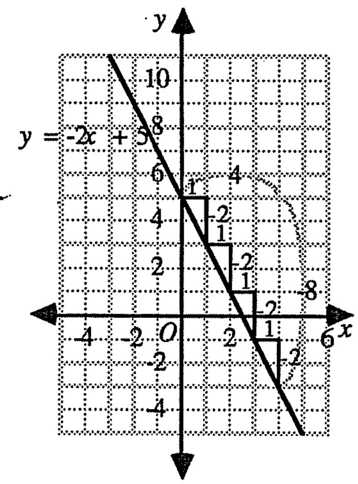
Problem 7 Check that this is true on the graph to the right.



If we consider the graph of the linear function $y = -2x + 5$ in the same way, we find that since its rate of change is -2, the line moves down 2 squares for every 1 square it moves to the right.

In general, the line moves down $2h$ when it moves h to the right.

Problem 8 Check that this is true on the graph to the right.

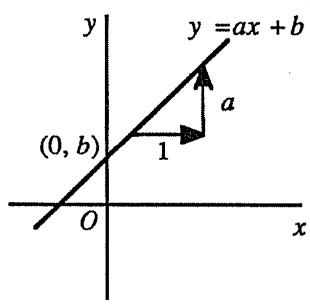


As you can now see, the inclination of the graph of the linear function $y = ax + b$ is determined by a. We call a the **slope** of this graph.

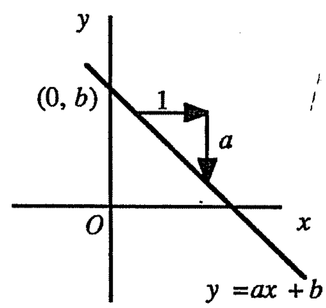
> **The Graph of a Linear Function**
>
> The graph of a linear function $y = ax + b$ is a straight line with a slope a and an intercept b.

The slope a is the increase in y when x increases by an increment of 1.

for $a > 0$



for $a < 0$



The graph of $y = \frac{1}{2}x + 3$ is a straight line with a slope of $\frac{1}{2}$ and an intercept of 3. This graph is referred to as the line $y = \frac{1}{2}x + 3$, and $y = \frac{1}{2}x + 3$ is said to be the equation of this line.

Problem 9 Given the following linear functions, find the slopes and intercepts of their graphs.

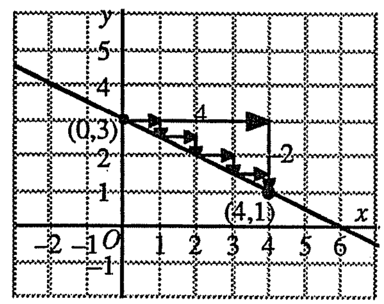
(1) $y = x - 2$ (2) $y = 3x - 4$

(3) $y = \frac{3}{2}x + 6$ (4) $y = -4x + \frac{2}{3}$

Because the graphs of linear functions are straight lines, we can draw them by using their slopes and intercepts.

Example 1 Graph the linear function $y = -\frac{1}{2}x + 3$.

[Solution] The intercept is 3, so the graph passes through the point (0, 3) on the y-axis. The slope is $-\frac{1}{2}$, so that the point (4, 1), for example, which is 4 squares to the right and 2 squares down from the point (0, 3) also lies on this graph. Therefore, if we draw a straight line through the two points (0, 3), (4, 1), we obtain the graph of this function.



Problem 10 Graph the following sets of linear functions, using the same coordinate axes for each set.

(1) (a) $y = 2x + 2$
(b) $y = 2x - 2$
(c) $y = 2x + 5$

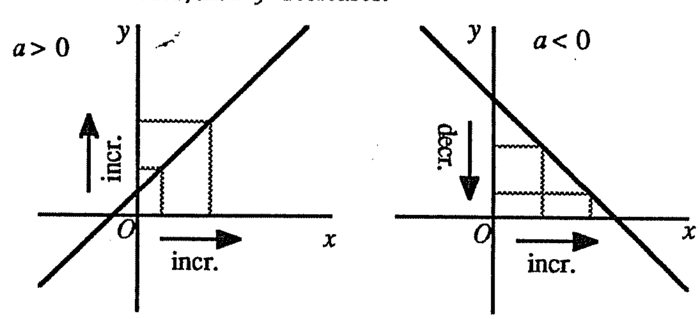
(2) (a) $y = 2.5x + 3$
(b) $y = -4x + 3$
(c) $y = -\frac{3}{2}x + 3$

Graphs and Changing Values in Linear Functions

In the linear function $y = ax + b$,

(1) For $a > 0$,
the graph is a straight line rising to the right, and if x increases, then y also increases.

(2) For $a < 0$,
the graph is a straight line falling to the right, and if x increases, then y decreases.



incr. = increasing
decr. = decreasing

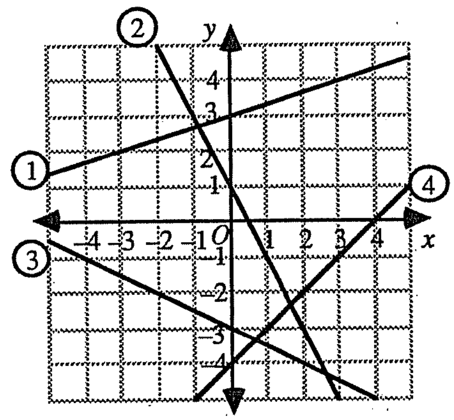
Problem 11 Graph the following linear functions using the same coordinate axes. Which lines are parallel?

(1) $y = -x - 2$ (2) $y = \frac{1}{3}x + 2$

(3) $y = -3x$ (4) $y = -x + 3$

(5) $y = \frac{1}{3}x - 3$ (6) $y = -3x + 4$

Problem 12 Find the equations for lines (1)–(4) in the diagram below.



4 Using Linear Functions to Establish Corresponding Values

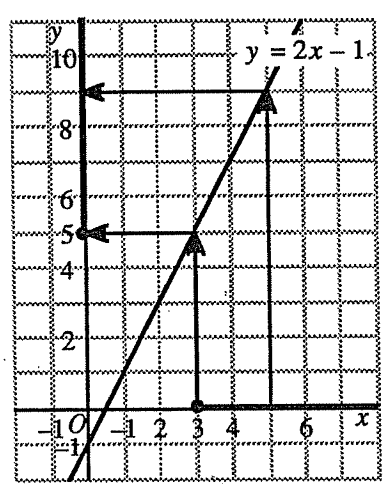
In the linear function

$$y = 2x - 1,$$

if $x = 3$, then

$$y = 2 \times 3 - 1 = 5$$

If $x \geq 3$, the values of x fall on the bold line along the x-axis, and the corresponding values of y lie on the bold line along the y-axis. The y-coordinate of the endpoint of the bold line along the y-axis is 5, which we found above. So if the range of values of x in this function is $x \geq 3$, then the range of values of y is $y \geq 5$.



Problem 1 Given the linear function $y = \frac{1}{2}x + 4$.

(1) Graph the function.

(2) Find the value of y that corresponds to $x = 6$.

(3) Find the range of values of y for $x \geq 6$.

Problem 2 Given the linear function $y = 3x - 2$.

(1) Graph the function.

(2) Find the values of y that correspond to $x = 2$ and $x = 3$.

(3) Find the range of values of y for $2 < x < 3$.

Example 1 Given $y = -2x + 3$, find the value of x that corresponds to $y = 5$.

[Solution] The x we want to find is the solution to the linear equation

$$5 = -2x + 3$$

If we solve this equation, we find

$$x = -1$$

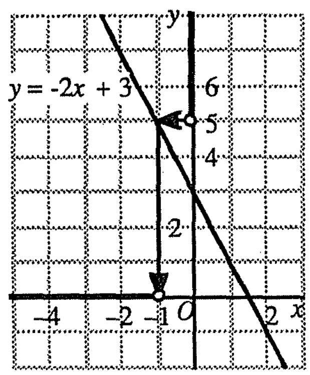
Answer: $x = -1$

Example 2 In the linear function in Example 1, what values can x take on for $y > 5$?

[Solution] The value of x that corresponds to $y = 5$ is $x = -1$. If we graph $y = -2x + 3$, we obtain the figure to the right. The values of y for $y > 5$ lie on the bold line along the y-axis, and the corresponding values of x lie on the bold line along the x-axis.

Therefore,

when $y > 5$, $x < -1$



Answer: $x < -1$

[Alternate Solution] To find the range of x, we can solve the linear inequality

$$-2x + 3 > 5$$

We find that

$$x < -1$$

Answer: $x < -1$

Problem 3 Given: $y = \frac{1}{3}x + 2$.

(1) Find the value of x that corresponds to $y = 3$.

(2) What values can x take for $y < 3$? As in Example 2, solve this problem in two ways: by graph and by calculation.

5 Creating Linear Functions

Let us now consider how to create a linear function of the form $y = ax + b$ which satisfies certain given conditions.

Given the Rate of Change and a Pair of Values x, y

Example 1 Given that the rate of change is $-\frac{1}{2}$ and $x = 3$, make up a linear function that will give us $y = -2$.

[Solution] Because the rate of change is $-\frac{1}{2}$, the function must take the form

$$y = -\frac{1}{2}x + b$$

The function takes on the value $y = 2$ for $x = 3$, so we substitute these values in the above equation, which gives us

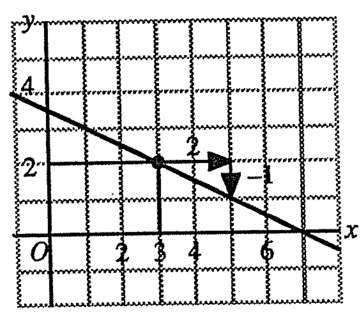
$$2 = -\frac{1}{2} \times 3 + b$$

Solving the equation, $b = \frac{7}{2}$

Therefore, the linear function we need is

$$y = -\frac{1}{2}x + \frac{7}{2}$$

Answer: $y = -\frac{1}{2}x + \frac{7}{2}$



Example 1 is equivalent to finding the equation for the straight line with a slope of $-\frac{1}{2}$ that passes through the point (3, 2).

Problem 1 Write a linear function to satisfy the following conditions.

(1) The rate of change is $\frac{1}{2}$, and $y = -2$ for $x = 0$.

(2) The rate of change is -1, and $y = -3$ for $x = -2$.

Problem 2 Write an equation for the straight line that satisfies the following conditions.

(1) The slope is -2.5, and the y-intercept is (0, 2).

(2) The line passes through the point (1, 3), and the slope is -2.

Given Two Pairs of Values for x, y

Example 2 Write a linear function that takes on the values $y = 4$ for $x = 3$, and $y = 10$ for $x = 12$.

[Solution] We need to find a linear function of the form $y = ax + b$.

Because $y = 4$ when $x = 3$,

$$4 = 3a + b \qquad (1)$$

Because $y = 10$ when $x = 12$,

$$10 = 12a + b \qquad (2)$$

We can solve (1) and (2) as simultaneous equations to find the values of a and b:

$$a = \frac{2}{3}, \quad b = 2$$

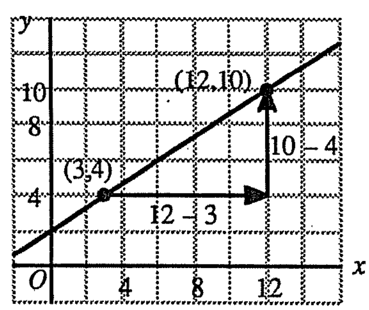
Therefore, the linear function we need is

$$y = \frac{2}{3}x + 2$$

Answer: $y = \frac{2}{3}x + 2$

[Alternate Solution] The slope of the straight line that passes through the two points (3, 4) and (12, 10) is

$$\frac{10 - 4}{12 - 3} = \frac{2}{3}$$



Therefore,

$$y = \frac{2}{3}x + b$$

If we substitute $x = 3$ and $y = 4$ into the original equation and then solve for b, we find that

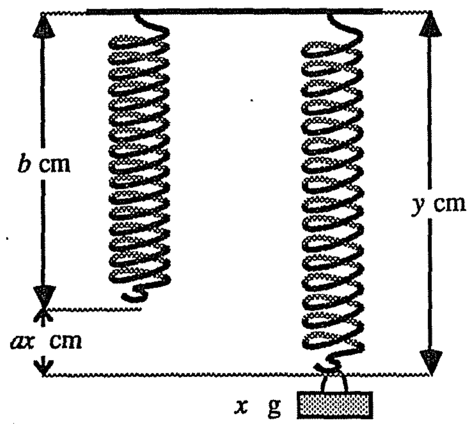
$$b = 2$$

Answer: $y = \frac{2}{3}x + 2$

Problem 3 Write a linear function that takes on the values $y = -3$ for $x = 2$, and $y = -9$ for $x = 4$.

Problem 4 Write the equation of a straight line that passes through the two points (-3, 5) and (2, -1).

Example 3 The expansion of a spring is proportional to the mass of a weight hanging from it. The length of the spring is 12 cm when a 10 g weight is attached to it, and it is 16 cm long when a 30 g weight is attached. Assuming that the length of the spring is y cm when a weight of x g hangs from it, express y in terms of x.



[Approach] If we assume that the length of the spring with no weight attached is b cm, and the expansion of the spring is a cm for each 1 g of mass, then y is expressed by the formula.

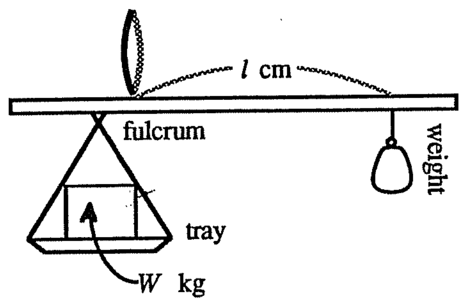
$$y = ax + b \qquad (1)$$

Problem 5 Substitute the values of x and y given in Example 3 into the formula in (1), and make up simultaneous equations for a, b. Solve the equations to find the values of a, b.

Problem 6 We have a candle that burns in such a way that the reduction in its length is proportional to the time it burns. The candle was 12 cm long 6 minutes after it started to burn, and 16 minutes later it was 7 cm long. If the length of the candle is y cm after it has burned x minutes, express y in terms of x, and draw the graph. How long will the candle burn before it is used up?

Establishing Functions Through Experiments and Observations

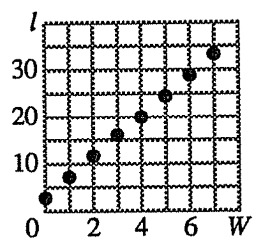
We put objects of various weights on a beam balance, and balance them by hanging a weight at varying distance from the fulcrum. The relation between the weight of each object, W kg, and the distance from the fulcrum to the weight, l cm, is given in the following table.

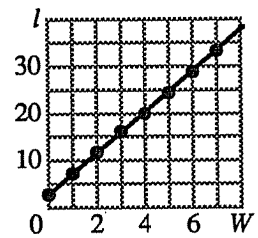


W	0	1	2	3	4	5	6	7
l	3.0	7.5	11.9	16.5	21.0	25.6	30.1	34.5

Applying these observations, let's derive an equation to represent the relation betwee W and l.

If we plot points whose coordinates are given by the paired values of W and l, the points will be arranged as in the diagram at the left below: approximately along a straigh line.





From the diagrams the intercept of this line is 3, and the slope is 4.5, so the equati of this line is

$$l = 4.5W + 3 \quad (1)$$

Problem 7 Calculate the value of l that corresponds to $W = 2, W = 5$ fro the equation in (1), and compare it to the values in the table above Calculate the value of l that corresponds to $W = 3.5, W = 8$.

Exercises

1. Given the linear function $y = -3x + 1$, find the values of y that correspond to $x = -1, x = \frac{1}{3}$.

2. Write equations to express the relations between x and y given in the following tables.

(1)

x	-3	0	3	6
y	-6.5	-5	-3.5	-2

(2)

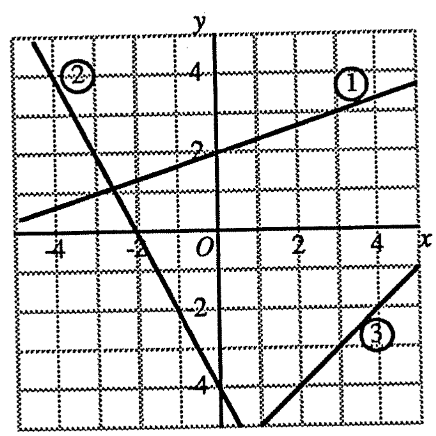
x	-1	1	3	5
y	2.5	-1.5	-5.5	-9.5

3. Graph the following linear functions, using the same coordinate axes.

(1) $y = 3x - 4$ (2) $y = \frac{1}{2}x$

(3) $y = -x + 4$ (4) $y = -2x + 1$

4. Write the equations of lines (1)–(3) in the diagram below.



5. In the linear function $y = ax + 4$, the increase in y is -3 when x increases by 2. Find the value of a.

6. Write linear functions to satisfy the following conditions.

(1) $y = 3$ for $x = 2$, and the rate of change is 0.5.

(2) $y = 3$ for $x = -3$, and $y = 5$ for $x = 3$.

(3) The graph passes through the point (-2, 3) and is parallel to the straight line $y = -x$.

2 LINEAR FUNCTIONS AND LINEAR EQUATIONS IN TWO VARIABLES

2 Graphs of Linear Equations in Two Variables

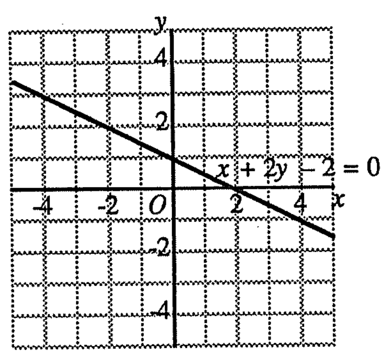
The following expression is a linear equation in two variables which includes two letters, x and y.

$$x + 2y - 2 = 0 \tag{1}$$

Problem 1 For the equation in (1), find the values of y that correspond to the following values of x, and fill in the blanks in the table below.

x	......	-3	-2	-1	0	1	2	3	......
y	......								......

An infinite number of paired values of x, y satisfy (1). If we draw a graph by plotting the points whose coordinates are given by the paired values of x, y, it will look like the one in the diagram below.



In equation (1), y is a function of x, since the value of y is established once we determine the value of x. It is easier to view equation (1) as a function if we solve for y:

$$y = -\frac{1}{2}x + 1 \tag{2}$$

The graph we drew above is simply the graph of the linear function in (2). It is a straight line with a slope of $-\frac{1}{2}$ and an intercept of 1.

This straight line is referred to as the **graph of the equation** $x + 2y - 2 = 0$.

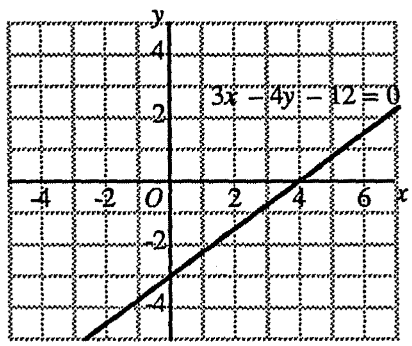
The graph of the equation $x + 2y - 2 = 0$ is the set of all the points whose coordinates are given by the paired values of x, y that satisfy the equation.

Example 1 Graph the equation $3x - 4y - 12 = 0$.

[Solution] Solving the equation for y,

$$y = \frac{3}{4}x - 3$$

Therefore, the graph we are looking for is a straight line with a slope of $\frac{3}{4}$ and an intercept of -3, as shown in the figure below.



Problem 2 Graph the following equations:

(1) $-x + 2y = 4$ (2) $3x - 2y + 8 = 0$

Next, let's graph a linear equation in two variables of the form $ax + bx + c = 0$, where the coefficient of x or y is 0.

Example 2 Let's graph $2y - 6 = 0$.

This expression is a linear equation in two variables of the form $0x + 2y - 6 = 0$. Therefore, regardless of the value of x, it remains

$$2y - 6 = 0,$$

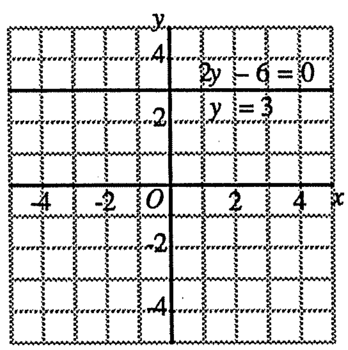
and so

$$y = 3.$$

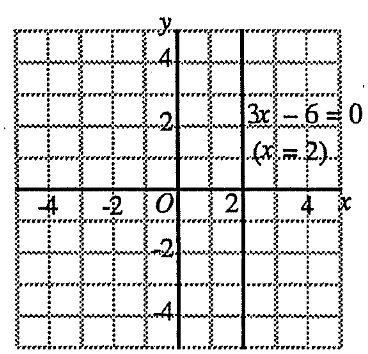
This graph is the set of all the points whose y-coordinate is 3. For example, the following paired values are points on this graph:

$$\ldots, (-1, 3), (0, 3), (1, 3), (2, 3), \ldots$$

Therefore, the graph is a straight line passing through the point (0, 3) parallel to the x-axis.



Example 3 Let's graph $3x - 6 = 0$.



This expression is a linear equation in two variables of the form $3x + 0y - 6 = 0$. Therefore, regardless of the value of y, it remains

$$3x - 6 = 0,$$

and so

$$x = 2.$$

This graph is the set of all the points whose x-coordinate is 2, so it is a straight line which passes through the point (2, 0) parallel to the y-axis.

Problem 3 Graph the following equations:

(1) $5y = 10$ (2) $2x = -5$ (3) $-x + 3 = 0$

> **Graphs of Linear Equations in Two Variables**
>
> The graph of a linear equation in two variables of the form $ax + by + c = 0$, where a, b, and c are constants, is a straight line.
>
> In particular,
> if $a = 0$, the graph is a straight line parallel to the x-axis,
> and if $b = 0$, the graph is a straight line parallel to the y-axis.

Example 4 Graph the equation $2x - 3y + 6 = 0$.

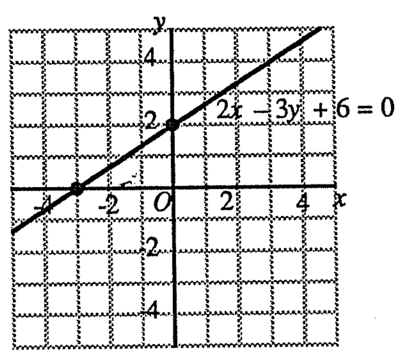
[Approach] Since the graph of $2x - 3y + 6 = 0$ is a straight line, we can graph it by finding two points through which the line passes.

[Solution]

If $x = 0$, then $y = 2$;
if $y = 0$, then $x = -3$.

Therefore, the graph is a straight line passing through the two points:

(0, 2) and (-3, 0)



Problem 4 Graph the following equations:

(1) $2x - y = 1$ (2) $x + y = 2$ (3) $4x + 3y - 6 = 0$

② Solving Simultaneous Linear Equations in Two Variables by Graphing Them

In the preceding chapter, we learned to solve simultaneous equations by calculating. For example, the solution to

$$\begin{cases} x + y = 8 & (1) \\ 2x - y = 1 & (2) \end{cases}$$

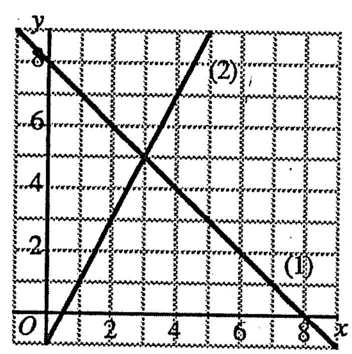
is $x = 3$, $y = 5$.

Problem 1 Solve the simultaneous equations above and check this solution.

Let's examine the relation between the solution to these equations and a graph of (1) and (2).

The set of all the points whose coordinates are given by the paired values of x, y which satisfy equation (1) is the straight line (1) in the diagram to the right.

Similarly, the set of all the points whose coordinates are given by the paired values of x, y which satisfy equation (2) is the straight line (2) in the diagram to the right.



Thus, the values of x and y which satisfy both (1) and (2) belong to the point whose coordinates are the solution to the simultaneous equations (1) and (2), i.e., the intersection of line (1) and line (2).

> **Solutions to Simultaneous Equations and Intersections of Graphs**
>
> The solution to simultaneous equations in x and y is given by the paired coordinates x, y of the point at which the graphs of the equations intersect.

Problem 2 Read the coordinates of the intersection of the two lines in the diagram on the preceding page, and check whether the *x*- and *y*-coordinates of that point are the solution to the simultaneous equations on the preceding page.

Problem 3 Solve the following simultaneous equations by graphing them.

(1) $\begin{cases} 3x + y = 2 \\ 2x - y = 3 \end{cases}$ (2) $\begin{cases} 3x - y = -4 \\ 2x - y = 0 \end{cases}$ (3) $\begin{cases} y = 3x - 2 \\ y = 4 \end{cases}$

Problem 4 A road runs alongside a railroad. When a train traveling at 0.8 km/minute left station A at 7:00, there was a bus 1.5 km ahead moving at a speed of 0.5 km/minute. Assuming that the train and the bus are y km from station A at x minutes after 7:00, express the relation between x and y in equations for each vehicle. Graph both equations on the same coordinate axes, and find the time when the train will catch up with the bus.

Exercises

1. Graph the following equations on the same coordinate axes:

(1) $2x + y = 7$ (2) $3x - y = 3$

(3) $2x - y - 8 = 0$ (4) $3x + y - 7 = 0$

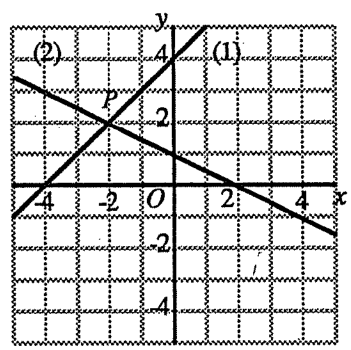
2. Use the graphs you drew in problem 1 to solve the following simultaneous equations: (1) and (2); (3) and (4).

3. Given two straight lines

$$y = x + 4 \qquad (1)$$

$$y = -\frac{1}{2}x + 1 \qquad (2)$$

on a graph whose unit is 1 cm,



(1) Find the coordinates of point P, the intersection of lines (1) and (2).

(2) Find the length of segment AB, if the straight line $x = 1$ intersects lines (1) and (2) at points A and B, respectively.

Chapter Exercises

A

1. Given the linear function $y = 2(x + 1)$.

   (1) Find the value of y that corresponds to $x = -3, x = 2$.

   (2) Find the range of the values of y for $-3 \leq x \leq 2$.

   (3) Find the increase in y if the value of x increases by 3.

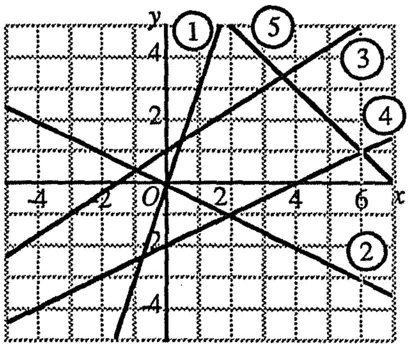
2. Find the linear function that satisfies the following conditions.

   (1) $y = 3$ for $x = 5$, and y increases by 2 if x increases by 5.

   (2) The graph passes through the two points (2, 3) and (-5, -11).

   (3) The graph passes through the point (1, -2) and is parallel to the straight line $y = -3x - 5$.

3. Find an equation for the linear function whose graph is the straight line in the diagram to the right. Calculate the coordinates of the points at which (1) intersects (4) and (3) intersects (5).



4. The relation between x and y is expressed by the equation $y = -2x + 3$.

   (1) Find the value of x which corresponds to $y = 5$.

   (2) If $y \geq 5$, what value can x take on ?

5. Given the linear function $y = -\frac{1}{2}x + 5$, find the range of values of y for $-2 \leq x \leq 6$.

6. Find the equations for the straight lines parallel to the x-axis and parallel to the y-axis which pass through the intersection of the lines $2x + y = -1$ and $x - 3y = -4$.

7. A souvenir photograph will cost 500 yen for a set of 3 copies. Additional copies will cost 70 yen apiece. Assuming that the total price is y yen when we order x additional copies, express y in terms of x. Find the total price if we order 15 additional copies.

8. Let the temperature at a point x km above the ground be y °C when the temperature on the ground is 15°C. If the value of x ranges from $0 \le x \le 10$, the approximate value of y can be expressed by the equation

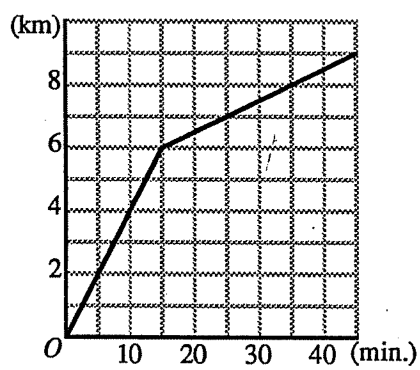
$$y = -6x + 15$$

(1) Within the above range, how many degrees does the temperature fall for each 1 km increase in altitude?

(2) What is the temperature at 2 km and 5 km above the ground?

(3) What is the range in altitude if the temperature range is from 15°C to 0°C inclusively?

9. A man left his house at 8:00 a.m., rode to town A on a bicycle, and then walked from town A to town B. The graph to the right shows the relation between x and y, if the man is y km from his house x minutes after leaving the house.



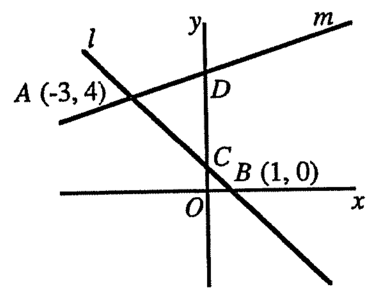
(1) Write an equation to express the relation between x and y from the time the man left his house until he arrived at town A.

(2) Find the speed in m/minute at which he walked from town A to town B.

(3) 15 minutes after he left the house, his older brother left for town B on a bicycle at a speed of 42 km/hour. By drawing a graph, find the time at which the brother would catch up with him. At what distance from their house would they meet?

B

1. In the diagram to the right, l is a straight line passing through point $A(-3, 4)$ and point $B(1, 0)$ on the x-axis, and m is a straight line passing through point A with a slope of $\frac{1}{3}$.

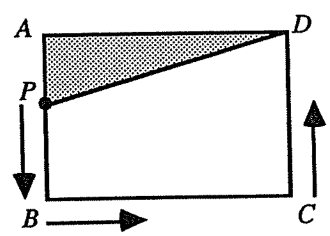


   (1) When the x-coordinate of a point on line l changes from -2 to 3, how does the corresponding y-coordinate change?

   (2) Find the coordinates of point D, at which line m intersects the y-axis.

   (3) Find the coordinates of the intersection of lines l and m', if the intercept of m', parallel to line m, is 3.

2. The rectangle to the right is 6 cm wide and 10 cm long. Point P starts out from point A and moves at a speed of 2 cm/second along the perimeter of the rectangle, passing B and C and finally reaching D. Assume that the area of triangle APD is y cm^2 x seconds after P leaves A.



   (1) What range of values can x take as point P moves along CD?

   (2) Express y in terms of x as point P moves along CD and graph the resulting equation.

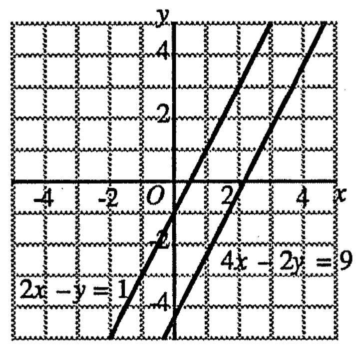
Advanced Topics for Individual Study

Simultaneous Linear Equations in Two Variables with No Solution

If we try to solve the simultaneous equations

$$\begin{cases} 2x - y = 1 & (1) \\ 4x - 2y = 9 & (2) \end{cases}$$

by graphing them, the graphs of (1) and (2) are parallel straight lines; they have no intersection, as we see in the diagram to the right. Therefore, there are no paired values of x, y that satisfy both (1) and (2), and these simultaneous equations have no solution.

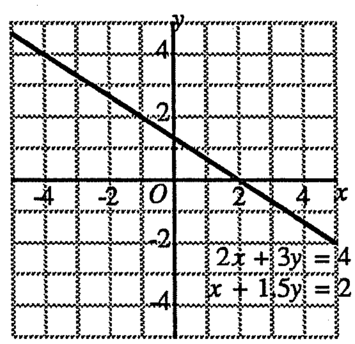


Simultaneous Linear Equations in Two Variables with an Infinite Number of Solutions

If we try to solve the simultaneous equations

$$\begin{cases} 2x + 3y = 4 & (3) \\ x + 1.5y = 2 & (4) \end{cases}$$

by graphing them, the graphs of (3) and (4) are the same straight line, as we see in the diagram to the right. Because the pairs of x- and y-coordinates of the points on the line are all solutions to both (3) and (4), these simultaneous equations have an infinite number of solutions.



Problem 1 Which of the following simultaneous equations have no solution? Which have an infinite number of solutions? Answer these questions by graphing the equations.

(1) $\begin{cases} 2x + y = 3 \\ 5x + 2.5y = 7.5 \end{cases}$ (2) $\begin{cases} 3x - y = 1 \\ x - \frac{1}{3}y = \frac{2}{3} \end{cases}$